

ME 5550 Intermediate Dynamics

Angular Momentum and Kinetic Energy of a Simple Crank Shaft

ME 5550 – Example #11

The figure shows a *simple crank shaft* consisting of *seven segments*, each considered to be a *slender bar*. Each segment of *length* ℓ has *mass* m . There are six segments of length ℓ and one segment of length 2ℓ (segment 4). The mass center of the system is G and is located on the axis of rotation.

Reference frames:

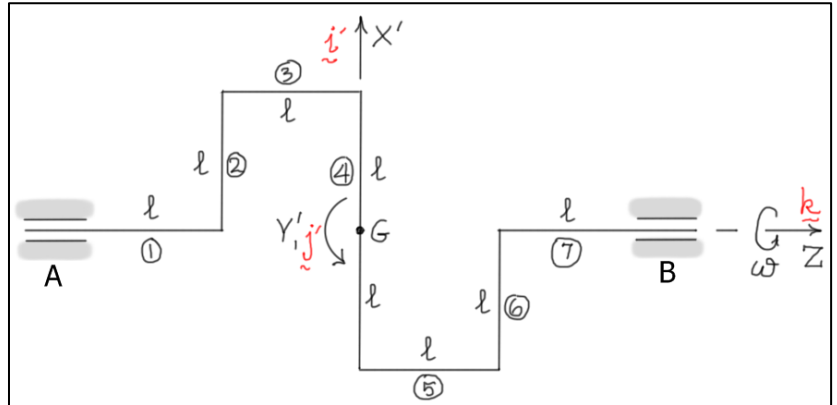
R : $\underline{i}, \underline{j}, \underline{k}$ (fixed frame)

S : $\underline{i}', \underline{j}', \underline{k}$ (rotates with the shaft)

Find:

\underline{H}_G ... angular momentum of the system about its mass center

K ... kinetic energy of the system



Solution: (resolving the inertia and angular velocity about the shaft-fixed axes)

$$\begin{Bmatrix} \underline{H}_G \cdot \underline{i}' \\ \underline{H}_G \cdot \underline{j}' \\ \underline{H}_G \cdot \underline{k} \end{Bmatrix} = \begin{bmatrix} I_{X'X'}^G & -I_{X'Y'}^G & -I_{X'Z}^G \\ -I_{Y'X'}^G & I_{Y'Y'}^G & -I_{Y'Z}^G \\ -I_{ZX'}^G & -I_{ZY'}^G & I_{ZZ}^G \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} = \begin{Bmatrix} -I_{X'Z}^G \omega \\ -I_{Y'Z}^G \omega \\ I_{ZZ}^G \omega \end{Bmatrix} \quad (1)$$

where

$$I_{ZZ}^G = \sum_{i=1}^7 (I_{ZZ}^G)_i = 0 + \frac{1}{3}m\ell^2 + m\ell^2 + \frac{1}{12}(2m)(2\ell)^2 + m\ell^2 + \frac{1}{3}m\ell^2 + 0 = \boxed{\frac{10}{3}m\ell^2}$$

$$I_{X'Z}^G = \sum_{i=1}^7 (I_{X'Z}^G)_i = 0 + m\left(\frac{\ell}{2}\right)(-\ell) + m(\ell)\left(-\frac{\ell}{2}\right) + 0 + m(-\ell)\left(\frac{\ell}{2}\right) + m\left(-\frac{\ell}{2}\right)(\ell) + 0 = \boxed{-2m\ell^2}$$

$$I_{Y'Z}^G \equiv 0 \quad (X'Z \text{ plane is a plane of symmetry, so products associated with } Y' \text{ are zero})$$

Substituting the inertias into Eq. (1) gives

$$\boxed{\underline{H}_G = 2m\ell^2 \omega \underline{i}' + \left(\frac{10}{3}\right)m\ell^2 \omega \underline{k}}$$

The *kinetic energy* of the crank shaft is found from the *velocity* and *angular momentum* vectors to be

$$K = \frac{1}{2} \underbrace{m(\underline{v}_G^R)^2}_{\text{zero}} + \frac{1}{2} \underline{\omega}_B^R \cdot \underline{H}_G = \frac{1}{2} \underline{\omega}_B^R \cdot \underline{H}_G = \frac{1}{2} (\omega \underline{k}) \cdot \underline{H}_G \Rightarrow \boxed{K = \frac{10}{6} m\ell^2 \omega^2}$$