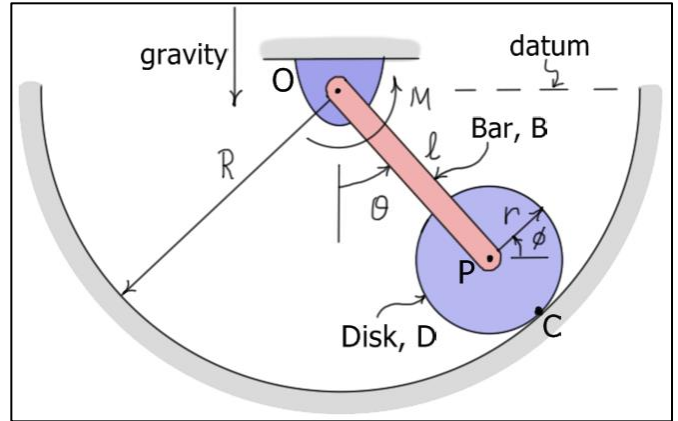


ME 5550 Intermediate Dynamics Lagrange's Equations Examples

Example #1

The system at the right consists of *two bodies*, a *slender bar B* and a *disk D*, moving together in a *vertical plane*. As *B* rotates about *O*, *D* rolls *without slipping* on the fixed circular outer surface. The length of *B* is ℓ , the radius of *D* is r , and the radius of the outer surface is R . The mass of the bar and disk are both m . The external torque $M(t)$ drives the system.



Equation of Motion

Using θ as the single *generalized coordinate*, the equation of motion of the system can be found from Lagrange's equation.

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = F_{\theta}} \quad (1)$$

Here, L is the Lagrangian of the system and is defined as follows.

$$\boxed{L = K - V = K_B + K_D - V_B - V_D}$$

The individual kinetic and potential energy terms are as follows.

$$K_D = \frac{1}{2} \omega_D \cdot \underline{H}_C = \frac{1}{2} I_C \dot{\phi}^2 = \frac{1}{2} \left(\frac{1}{2} m r^2 + m r^2 \right) \dot{\phi}^2 \Rightarrow \boxed{K_D = \frac{3}{4} m r^2 \dot{\phi}^2} \quad (\text{fixed axis rotation})$$

$$K_B = \frac{1}{2} \omega_B \cdot \underline{H}_O = \frac{1}{2} I_O \dot{\theta}^2 = \frac{1}{2} \left(\frac{1}{3} m \ell^2 \right) \dot{\theta}^2 \Rightarrow \boxed{K_B = \frac{1}{6} m \ell^2 \dot{\theta}^2} \quad (\text{fixed axis rotation})$$

$$V = V_D + V_B = -m g \ell C_{\theta} - \frac{1}{2} m g \ell C_{\theta} \Rightarrow \boxed{V = -\frac{3}{2} m g \ell C_{\theta}}$$

The concept of *instantaneous centers* can now be used to express L in terms of θ and $\dot{\theta}$ only. Specifically, the velocity of point P can be calculated as a point at the end of B or at the center of D . That is, $\boxed{v_P = \ell \dot{\theta} = -r \dot{\phi}}$. Using this equation to remove $\dot{\phi}$ from the Lagrangian gives

$$\boxed{L = \frac{11}{12} m \ell^2 \dot{\theta}^2 + \frac{3}{2} m g \ell C_{\theta}}$$

The **generalized active force** F_θ and the derivatives of the Lagrangian are as follows.

$$F_\theta = M \underline{\hat{k}} \cdot \frac{\partial}{\partial \dot{\theta}} (\underline{\omega}_B) = M \underline{\hat{k}} \cdot \underline{\hat{k}} = M(t)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{11}{6} m \ell^2 \dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{11}{6} m \ell^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{3}{2} m g \ell S_\theta$$

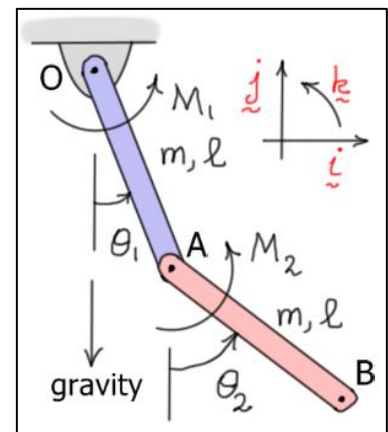
Substituting these results into Lagrange's Eq. (1) gives the **equation of motion**.

$$\frac{11}{6} m \ell^2 \ddot{\theta} + \frac{3}{2} m g \ell S_\theta = M(t) \quad (2)$$

Eq. (2) is a **nonlinear, second-order, ordinary differential equation**.

Example #2 – Double Pendulum

The figure to the right shows a **double pendulum** in a vertical plane with **driving torques** M_1 and M_2 at the connecting joints. The two **uniform slender links** are assumed to be identical with mass m and length ℓ . The system has **two degrees-of-freedom** described by the **generalized coordinate set** (θ_1, θ_2) .



Equation of Motion

Using θ_1 and θ_2 as the two **generalized coordinates**, the **equations of motion** of the system can be found using **Lagrange's equations**.

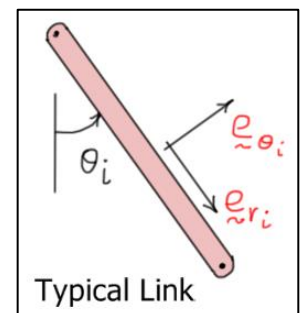
$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} &= F_{\theta_1} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} &= F_{\theta_2} \end{aligned} \quad (3)$$

Kinematics

The velocities and squares of velocities of the mass centers of the two links can be found using the concept of **relative velocity** as follows. For the definition of the unit vectors, refer to the diagram.

$$\underline{v}_{G_1} = \underline{v}_O + \underline{v}_{G_1/O} = \frac{1}{2} \ell \dot{\theta}_1 \underline{e}_{\theta_1}$$

$$\underline{v}_{G_2} = \underline{v}_A + \underline{v}_{G_2/A} = \underline{v}_O + \underline{v}_{A/O} + \underline{v}_{G_2/A} = \ell \dot{\theta}_1 \underline{e}_{\theta_1} + \frac{1}{2} \ell \dot{\theta}_2 \underline{e}_{\theta_2}$$



$$v_{G_1}^2 = \underline{v}_{G_1} \cdot \underline{v}_{G_1} = \frac{1}{4} \ell^2 \dot{\theta}_1^2$$

$$v_{G_2}^2 = \underline{v}_{G_2} \cdot \underline{v}_{G_2} = \ell^2 \dot{\theta}_1^2 + \frac{1}{4} \ell^2 \dot{\theta}_2^2 + 2 \left(\frac{1}{2} \ell^2 \dot{\theta}_1 \dot{\theta}_2 \right) (\underline{e}_{\theta_1} \cdot \underline{e}_{\theta_2}) = \ell^2 \dot{\theta}_1^2 + \frac{1}{4} \ell^2 \dot{\theta}_2^2 + \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1}$$

Here, C_{2-1} represents the cosine of $\theta_2 - \theta_1$.

Kinetic Energy

The **kinetic energy** of the system can be written as the sum of the kinetic energies of the two links.

$$K = K_1 + K_2$$

Here,

$$K_1 = \frac{1}{2} I_O \dot{\theta}_1^2 = \frac{1}{2} \left(\frac{1}{3} m \ell^2 \right) \dot{\theta}_1^2 = \frac{1}{6} m \ell^2 \dot{\theta}_1^2 \quad (\text{fixed axis rotation})$$

$$K_2 = \frac{1}{2} m v_{G_2}^2 + \frac{1}{2} I_{G_2} \dot{\theta}_2^2 = \frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{8} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} + \frac{1}{24} m \ell^2 \dot{\theta}_2^2$$

$$\Rightarrow K_2 = \frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{6} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} \quad (\text{general plane motion})$$

Potential Energy

Assuming the horizontal **datum** is level with the point O , the **potential energy** of the system can be written as follows.

$$V = V_1 + V_2 = -\frac{1}{2} m g \ell C_1 - m g \left(\ell C_1 + \frac{1}{2} \ell C_2 \right) = -\frac{3}{2} m g \ell C_1 - \frac{1}{2} m g \ell C_2$$

Lagrangian $L \triangleq K - V$

Using the above results, the Lagrangian can be written as

$$L = \frac{2}{3} m \ell^2 \dot{\theta}_1^2 + \frac{1}{6} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} + \frac{3}{2} m g \ell C_1 + \frac{1}{2} m g \ell C_2 \quad (4)$$

Generalized Forces

The **generalized active forces** associated with the **driving torques** can be calculated as follows. Note that a torque of M_2 is applied to link 2 and a **reaction torque** of $-M_2$ is applied to link 1.

$$F_{\theta_1} = \left(M_1 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(-M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_2}{\partial \dot{\theta}_1} \right) \Rightarrow F_{\theta_1} = M_1 - M_2$$

$$F_{\theta_2} = \left(M_1 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(-M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(M_2 \underline{k} \cdot \frac{\partial \underline{\omega}_2}{\partial \dot{\theta}_2} \right) \Rightarrow F_{\theta_2} = M_2$$

Derivatives of Lagrangian

Using the expression given in Eq. (4), the derivatives of the Lagrangian can be calculated as follows.

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{4}{3}m\ell^2\dot{\theta}_1 + \frac{1}{2}m\ell^2\dot{\theta}_2 C_{2-1} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = \frac{4}{3}m\ell^2\ddot{\theta}_1 + \frac{1}{2}m\ell^2 C_{2-1}\ddot{\theta}_2 - \frac{1}{2}m\ell^2\dot{\theta}_2(\dot{\theta}_2 - \dot{\theta}_1)S_{2-1}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2}m\ell^2 C_{2-1}\dot{\theta}_1 + \frac{1}{3}m\ell^2\dot{\theta}_2 \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = \frac{1}{2}m\ell^2 C_{2-1}\ddot{\theta}_1 + \frac{1}{3}m\ell^2\ddot{\theta}_2 - \frac{1}{2}m\ell^2\dot{\theta}_1(\dot{\theta}_2 - \dot{\theta}_1)S_{2-1}$$

$$\frac{\partial L}{\partial \theta_1} = \frac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 S_{2-1} - \frac{3}{2}mg\ell S_1$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 S_{2-1} - \frac{1}{2}mg\ell S_2$$

Substituting into Lagrange's equations of Eq. (3) gives the following *equations of motion*.

$$\left(\frac{4}{3}m\ell^2 \right) \ddot{\theta}_1 + \left(\frac{1}{2}m\ell^2 C_{2-1} \right) \ddot{\theta}_2 - \left(\frac{1}{2}m\ell^2 S_{2-1} \right) \dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 = M_1(t) - M_2(t) \quad (5)$$

$$\left(\frac{1}{2}m\ell^2 C_{2-1} \right) \ddot{\theta}_1 + \left(\frac{1}{3}m\ell^2 \right) \ddot{\theta}_2 + \left(\frac{1}{2}m\ell^2 S_{2-1} \right) \dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 = M_2(t) \quad (6)$$

Together, Eqs. (5) and (6) represent a *coupled* set of *nonlinear*, *second-order*, *ordinary differential equations* of motion for the *double pendulum* with *driving torques*.

Example – Double Pendulum with Springs and Dampers

The figure at the right shows a *double pendulum* as in the above example with the *driving torques* replaced with a set of *torsional springs* and *dampers*. The equations of motion of this system are easily found using the results from the previous example given that

$$M_1 = -k_1\theta_1 - c_1\dot{\theta}_1$$

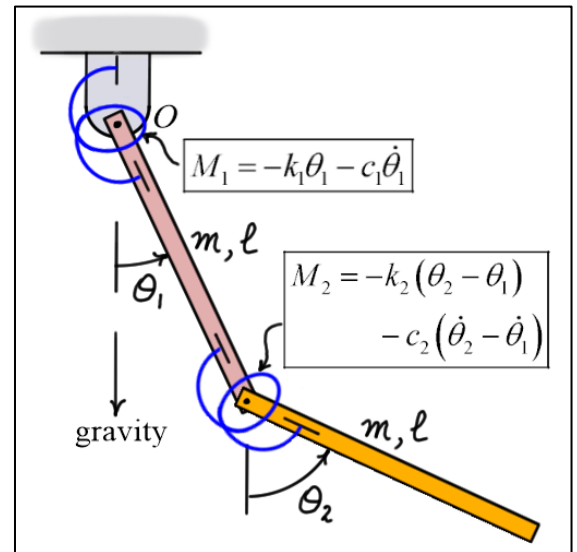
$$M_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

Substituting these results into Eqs. (5) and (6) gives

$$\left(\frac{4}{3}m\ell^2 \right) \ddot{\theta}_1 + \left(\frac{1}{2}m\ell^2 C_{2-1} \right) \ddot{\theta}_2 - \left(\frac{1}{2}m\ell^2 S_{2-1} \right) \dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 = -k_1\theta_1 - c_1\dot{\theta}_1 + k_2(\theta_2 - \theta_1) + c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

$$\left(\frac{1}{2}m\ell^2 C_{2-1} \right) \ddot{\theta}_1 + \left(\frac{1}{3}m\ell^2 \right) \ddot{\theta}_2 + \left(\frac{1}{2}m\ell^2 S_{2-1} \right) \dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 = -k_2(\theta_2 - \theta_1) - c_2(\dot{\theta}_2 - \dot{\theta}_1)$$

or



$$\boxed{\left(\frac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2 - \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0} \quad (7)$$

$$\boxed{\left(\frac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1 + \left(\frac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 + c_2(\dot{\theta}_2 - \dot{\theta}_1) + k_2(\theta_2 - \theta_1) = 0} \quad (8)$$

Eqs. (7) and (8) represent a set of *two simultaneous, nonlinear, second-order, ordinary differential equations of motion* of the double pendulum with springs and dampers at the connecting joints.