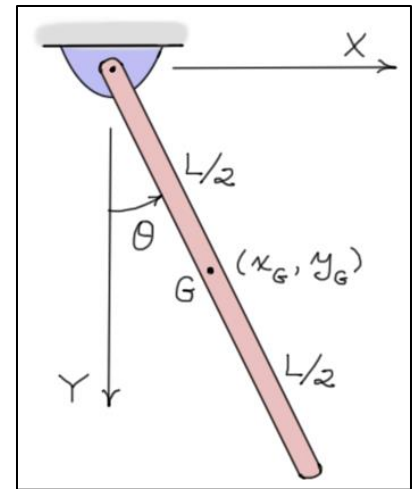


## ME 5550 Intermediate Dynamics

### Configuration Constraints for Mechanical Systems

Suppose the *configuration* of a mechanical system is defined by “ $n$ ” generalized coordinates, say  $q_k$  ( $k = 1, \dots, n$ ). These coordinates may all be *independent*, or they may form a *dependent* set. For example, consider the simple pendulum shown in the diagram. The coordinate set  $\{x_G, y_G, \theta\}$  is a *dependent* coordinate set. The following equations can be used to relate the coordinates.



$$\boxed{x_G = \frac{L}{2} \sin(\theta)} \quad \boxed{y_G = \frac{L}{2} \cos(\theta)}$$

Hence, for this system only *one* generalized coordinate is required to define its position. Any set of *two* or *more* coordinates forms a *dependent* set.

The types of constraints described above are referred to as *configuration constraints*. In general, for a mechanical system described by “ $n$ ” generalized coordinates  $q_k$  ( $k = 1, \dots, n$ ) with “ $m$ ” configuration constraints, the constraints can be written as

$$\boxed{f_j(q_1, q_2, \dots, q_n, t) = 0} \quad (j = 1, \dots, m)$$

Here, the constraints may also be dependent on the time,  $t$ . Configuration constraints are most useful in a dynamic analysis when they are differentiated into a form that is *linear* in the time derivatives of the coordinates.

$$\boxed{\frac{df_j}{dt} = \sum_{k=1}^n \left( \frac{\partial f_j}{\partial q_k} \right) \dot{q}_k + \frac{\partial f_j}{\partial t} = 0} \quad (j = 1, \dots, m)$$

or

$$\boxed{\sum_{k=1}^n a_{jk} \dot{q}_k + a_{j0} = 0} \quad (j = 1, \dots, m)$$

## Examples

1. For the simple pendulum shown above, define the generalized coordinate set as  $\{q_1, q_2, q_3\} = \{x_G, y_G, \theta\}$ . Then, the two configuration constraints are as follows.

$$\boxed{\begin{cases} f_1(q_1, q_2, q_3) \\ f_2(q_1, q_2, q_3) \end{cases} = \begin{cases} q_1 - \frac{L}{2} \sin(q_3) \\ q_2 - \frac{L}{2} \cos(q_3) \end{cases} = \begin{cases} 0 \\ 0 \end{cases}}$$

Differentiating these equations gives

$$\boxed{\begin{cases} \dot{q}_1 - \frac{L}{2} \dot{q}_3 \cos(q_3) \\ \dot{q}_2 + \frac{L}{2} \dot{q}_3 \sin(q_3) \end{cases} = \begin{cases} 0 \\ 0 \end{cases}} \Rightarrow \boxed{\begin{bmatrix} 1 & 0 & -\frac{L}{2} \cos(q_3) \\ 0 & 1 & +\frac{L}{2} \sin(q_3) \end{bmatrix} \begin{cases} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{cases} = \begin{cases} 0 \\ 0 \end{cases}}$$

Comparing these results to the general form shown above, we conclude

$$\boxed{a_{11} = a_{22} = 1}$$

$$\boxed{a_{12} = a_{21} = 0}$$

$$\boxed{a_{13} = -\frac{L}{2} \cos(\theta)}$$

$$\boxed{a_{23} = +\frac{L}{2} \sin(\theta)}$$

and

$$\boxed{a_{j0} = 0} \quad (j=1,2)$$

2. For the simple pendulum shown above, define the generalized coordinate set as  $\{q_1, q_2\} = \{x_G, y_G\}$ . Then, the configuration constraint is

$$\boxed{f(q_1, q_2) = q_1^2 + q_2^2 - (L/2)^2 = 0}$$

Differentiating the constraint gives

$$2q_1 \dot{q}_1 + 2q_2 \dot{q}_2 = 0 \quad \text{or} \quad \boxed{q_1 \dot{q}_1 + q_2 \dot{q}_2 = 0}$$

So, in this case,  $\boxed{a_{11} = q_1 = x_G}$ ,  $\boxed{a_{12} = q_2 = y_G}$ , and  $\boxed{a_{10} = 0}$ .