

ME 5550 Intermediate Dynamics

Constraint Relaxation Method: The Meaning of Lagrange Multipliers

Previously, it was noted that if a dynamic system is described using “ n ” *generalized coordinates* q_k ($k=1,\dots,n$), and if the system is subjected to “ m ” *independent configuration constraint equations* of the form

$$\boxed{\sum_{k=1}^n a_{jk} \dot{q}_k + a_{j0} = 0} \quad (j=1,\dots,m) \quad (1)$$

then, the equations of motion of the system can be found by using one of the following two forms of *Lagrange's equations* with *Lagrange multipliers*.

$$\boxed{\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^m \lambda_j a_{jk}} \quad (k=1,\dots,n) \quad (2)$$

or

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = (F_{q_k})_{nc} + \sum_{j=1}^m \lambda_j a_{jk}} \quad (k=1,\dots,n) \quad (3)$$

Here, K is the *kinetic energy* of the system, F_{q_k} is the *generalized force* associated with the generalized coordinate q_k , L is the *Lagrangian* of the system, V is the *potential energy function* for the *conservative forces* and *torques*, $(F_{q_k})_{nc}$ is the *generalized force* associated with q_k for the *non-conservative forces* and *torques*, only, λ_j is the *Lagrange multiplier* associated with the j^{th} *constraint equation*, and a_{jk} ($j=1,\dots,m$; $k=1,\dots,n$) are the *coefficients* from the *constraint equations*. Eqs. (1) along with Eqs. (2) or (3) form a set of “ $n+m$ ” *differential/algebraic equations* for the “ n ” *generalized coordinates* and the “ m ” *Lagrange multipliers*.

Alternatively, some or all the constraints can be *relaxed* (or *removed*), and they can be *replaced* with *force* and/or *torque components* that are required to maintain the constraints. Then, the “ n ” Lagrange’s equations can be formulated in terms of the “ n ” generalized coordinates and the “ m ” constraint force (and/or torque) components. Together with the constraint equations, this forms a *complete set* of “ $n+m$ ” *differential/algebraic equations* for the “ n ” generalized

coordinates and the “ m ” constraint force and/or torque components. If all the constraints are relaxed, then Eqs. (2) and (3) can be written as

$$\boxed{\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} = F_{q_k} + (F_{q_k})_{\text{constraints}}} \quad (k = 1, \dots, n) \quad (4)$$

and

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = (F_{q_k})_{nc} + (F_{q_k})_{\text{constraints}}} \quad (k = 1, \dots, n) \quad (5)$$

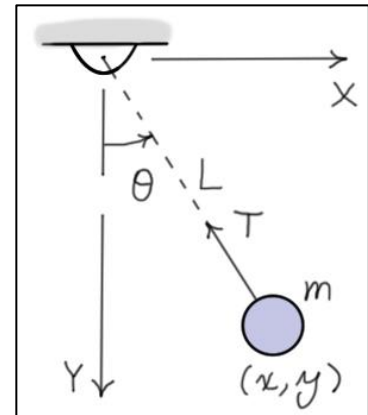
Comparing Eqs. (2) and (4) or Eqs. (3) and (5) gives the following.

$$\boxed{(F_{q_k})_{\text{constraints}} = \sum_{j=1}^m \lambda_j a_{jk}} \quad (k = 1, \dots, n) \quad (6)$$

So, the **Lagrange multipliers** are **directly related** to the **forces** and/or **torques** required to maintain the constraints.

Example: The Simple Pendulum

To illustrate constraint relaxation, consider the simple pendulum shown with $q_1 = x$ and $q_2 = y$ as the generalized coordinates. The fixed length of the pendulum can be used to **relate** the two coordinates, but here the **length constraint** will be **relaxed** in the formulation. Specifically, Lagrange’s equations can be written in the form



$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = (F_{q_k})_{nc} + (F_{q_k})_{\text{constraint}}} \quad (7)$$

Here, the Lagrangian $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy$, and as there are no nonconservative forces or

torques, $(F_{q_k})_{nc} = 0$. The contributions of the **constraint force** to the generalized forces can be

calculated as follows.

$$\boxed{(F_x)_{\text{constraint}} = \vec{T} \cdot (\partial \vec{y} / \partial \dot{x}) = T \left(-(x/L) \vec{i} - (y/L) \vec{j} \right) \cdot \left[\partial (\dot{x} \vec{i} + \dot{y} \vec{j}) / \partial \dot{x} \right] = -T(x/L)} \quad (8)$$

$$\boxed{\left(F_y\right)_{\text{constraint}} = T \cdot (\partial v / \partial \dot{y}) = T \left(-(x/L)\underline{i} - (y/L)\underline{j} \right) \cdot \left[\partial(\dot{x}\underline{i} + \dot{y}\underline{j}) / \partial \dot{y} \right] = -T(y/L)} \quad (9)$$

Substituting into Lagrange's equations (Eq. (7)) and supplementing with the twice-differentiated constraint equation it can be shown that the equations of motion can be written as follows.

$$\boxed{\begin{aligned} m\ddot{x} + \left(\frac{x}{L}\right)T &= 0 \\ m\ddot{y} - mg + \left(\frac{y}{L}\right)T &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \quad (10)$$

Using **Lagrange multipliers**, it was shown in previous notes that the equations for the pendulum could be written as

$$\boxed{\begin{aligned} m\ddot{x} - \lambda x &= 0 \\ m\ddot{y} - mg - \lambda y &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \quad (11)$$

Comparing Eqs. (10) and (11), the **Lagrange multiplier** λ is clearly equal to $-T/L$ the negative of the **force per unit length** of the **pendulum**.