ME 5550 Intermediate Dynamics
Lagrange's Equations – Example System II

In previous notes for Example System II, ${}^R\underset{\sim}{\omega}_B$ the ***angular velocity*** of the bar and $\underset{\sim}{H}_G$ the ***angular momentum*** of B resolved in ***bar-fixed*** directions $B:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{e}_3)$ were found to be

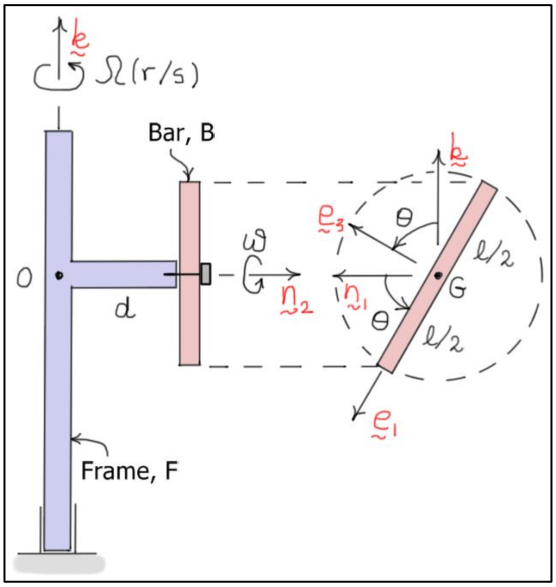
$$\boxed{{}^R\underset{\sim}{\omega}_B = (-\Omega S_\theta)\underset{\sim}{e}_1 + \omega \underset{\sim}{n}_2 + (\Omega C_\theta)\underset{\sim}{e}_3}$$

and

$$\boxed{\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B = \tfrac{m\ell^2}{12}\left[\omega \underset{\sim}{n}_2 + \Omega C_\theta \underset{\sim}{e}_3\right]}$$

Reference Frames

$$\boxed{\begin{array}{l} F:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{k}) \\ B:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{e}_3) \end{array}}$$



Here it is assumed that frame F is ***light*** and that torque $M_\phi(t)$ is applied to F by the ground and torque $M_\theta(t)$ is applied to B by F.

Assuming the degrees of freedom of the system are described by the ***generalized coordinates*** $\phi\ (\dot{\phi} = \Omega)$ and θ, the ***equations of motion*** of the system can be found using Lagrange's equations shown in Eq. (1).

$$\boxed{\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = F_\phi \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = F_\theta \end{array}} \tag{1}$$

<u>Lagrangian</u>

Assuming the horizontal ***datum*** is located at the level of the mass center G, the Lagrangian is simply the kinetic energy of the system.

$$L = K = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} \underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G \quad \Rightarrow \boxed{L = \tfrac{1}{2} m d^2 \dot{\phi}^2 + \tfrac{1}{24} m\ell^2\left(\dot{\theta}^2 + C_\theta^2 \dot{\phi}^2\right)} \tag{2}$$

<u>Generalized Forces</u>

The ***generalized forces*** associated with the ***driving torques*** can be calculated as follows. Note the torque M_θ is applied to B and the reaction torque $-M_\theta$ is applied to F.

$$F_\theta = \left(M_\theta \underset{\sim}{n}_2 \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\theta}} \right) + \left(-M_\theta \underset{\sim}{n}_2 \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_F}{\partial \dot{\theta}} \right) + \left(M_\phi \underset{\sim}{k} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_F}{\partial \dot{\theta}} \right) \quad \Rightarrow \boxed{F_\theta = M_\theta}$$

$$F_\phi = \left(M_\theta \underset{\sim}{n}_2 \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{\phi}} \right) + \left(-M_\theta \underset{\sim}{n}_2 \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_F}{\partial \dot{\phi}} \right) + \left(M_\phi \underset{\sim}{k} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_F}{\partial \dot{\phi}} \right) \quad \Rightarrow \boxed{F_\phi = M_\phi}$$

Derivatives of Lagrangian

Given the expression for the Lagrangian in Eq. (2), the derivatives of the Lagrangian can be calculated as follows.

$$\frac{\partial L}{\partial \dot{\phi}} = md^2\dot{\phi} + \tfrac{1}{12} m\ell^2 \dot{\phi} C_\theta^2 \qquad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \left(\tfrac{1}{12} m\ell^2 C_\theta^2 + md^2\right)\ddot{\phi} - \tfrac{1}{6} m\ell^2 \dot{\theta}\dot{\phi} S_\theta C_\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \tfrac{1}{12} m\ell^2 \dot{\theta} \qquad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \tfrac{1}{12} m\ell^2 \ddot{\theta}}$$

$$\boxed{\frac{\partial L}{\partial \phi} = 0} \qquad \boxed{\frac{\partial L}{\partial \theta} = -\tfrac{1}{12} m\ell^2 \dot{\phi}^2 S_\theta C_\theta}$$

Equations of Motion

Substituting the above results into Lagrange's equations (Eqs. (1)) gives

$$\boxed{\begin{aligned} \left(md^2 + \tfrac{1}{12} m\ell^2 C_\theta^2\right)\ddot{\phi} - \left(\tfrac{1}{6} m\ell^2 S_\theta C_\theta\right)\dot{\theta}\dot{\phi} &= M_\phi(t) \\ \left(\tfrac{1}{12} m\ell^2\right)\ddot{\theta} + \left(\tfrac{1}{12} m\ell^2 S_\theta C_\theta\right)\dot{\phi}^2 &= M_\theta(t) \end{aligned}} \tag{3}$$

Eqs. (3) represent a set of two ***coupled***, ***nonlinear***, ***second-order***, ***ordinary differential equations of motion***.

Ignorable Coordinates

When a ***generalized coordinate*** is ***missing*** from the Lagrangian (so the ***derivative*** of L with respect to that coordinate is ***zero***), the coordinate is said to be ***ignorable***. In the above example, if the driving torque $M_\phi(t)$ is zero, then the first of the Lagrange's equations reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \underbrace{\frac{\partial L}{\partial \phi}}_{\text{zero}} = 0 \quad \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0} \quad \Rightarrow \boxed{\frac{\partial L}{\partial \dot{\phi}} = md^2\dot{\phi} + \tfrac{1}{12} m\ell^2 \dot{\phi} C_\theta^2 = \text{constant}}$$

So, in the ***absence*** of other ***exciting forces*** or ***torques***, ignorable coordinates can be used to identify ***constants*** (integrals) of the system's motion.