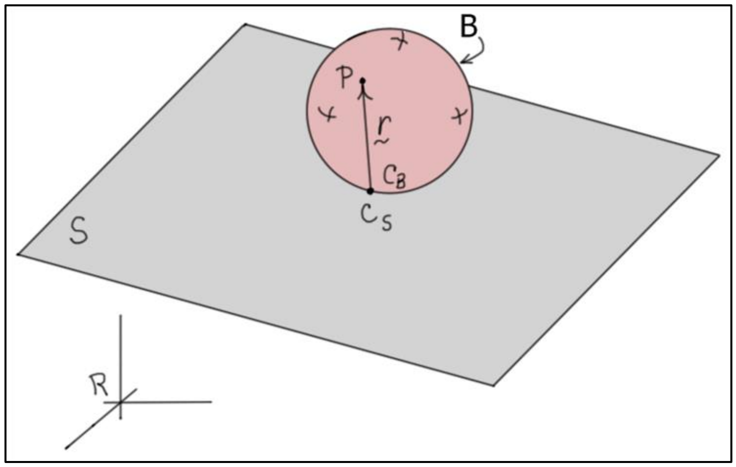
ME 5550 Intermediate Dynamics
Rolling Constraints – Point Contact

When a rigid body rolls (without slipping) on a rigid surface, the surface ***constrains*** its motion. Consider the body B rolling on the rigid surface S as shown in the diagram. Here,



- S : rigid surface
- B : rigid body
- C_B : contact point on B
- C_S : contact point on S
- R : fixed reference frame

The body B is said to ***roll*** (without slipping) on S if

$$\boxed{{}^{S}\underset{\sim}{v}_{C_B} = \underset{\sim}{0}} \quad \text{or} \quad \boxed{{}^{R}\underset{\sim}{v}_{C_B} = {}^{R}\underset{\sim}{v}_{C_S}}$$

The ***velocity*** of other points of B (e.g. P) can be found by using the formula for ***relative velocity***.

$${}^{R}\underset{\sim}{v}_{P} = {}^{R}\underset{\sim}{v}_{C_B} + {}^{R}\underset{\sim}{v}_{P/C_B}$$

or

$$\boxed{{}^{R}\underset{\sim}{v}_{P} = {}^{R}\underset{\sim}{v}_{C_S} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}_{P/C_B}\right)}$$

Direct differentiation can then be used to find the acceleration of point P. That is,

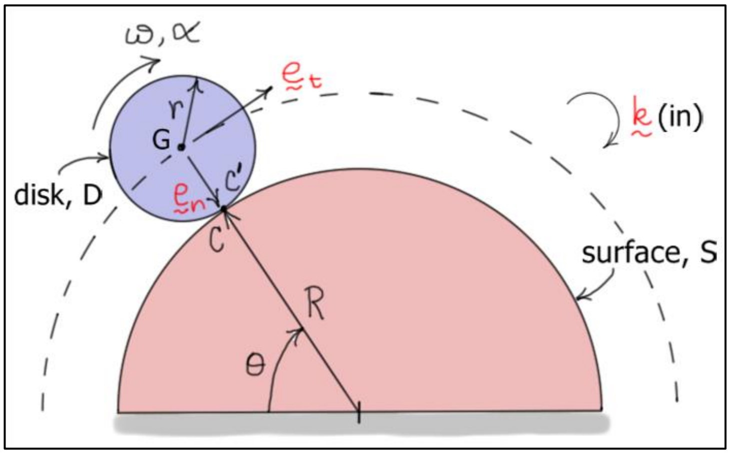
$$\boxed{{}^{R}\underset{\sim}{a}_{P} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P}\right)}$$

Note: Even though the velocities of the contact points C_B and C_S are ***equal*** (i.e. $\underset{\sim}{v}_{C_B} = \underset{\sim}{v}_{C_S}$), the accelerations of these points are ***not equal*** (i.e. $\underset{\sim}{a}_{C_B} \neq \underset{\sim}{a}_{C_S}$).

Rolling (without Slipping) in Two Dimensions

Rolling on a Fixed Surface

If a rigid body ***rolls*** (without slipping) on a ***fixed surface***, the point in ***contact*** with the surface has ***zero velocity***. For example, consider a ***circular*** disk D that rolls on the ***circular*** surface S as shown below.



Because C' is in ***contact*** with the point C on the ***fixed surface***, its velocity is ***zero***. Using this result, the velocity of G (center of the disk) can be calculated using the relative velocity equation.

$$\underset{\sim}{v}_G = \underset{\sim}{v}_{C'} + \underset{\sim}{v}_{G/C'} = \underset{\sim}{v}_{G/C'} = \underset{\sim}{\omega}_D \times \underset{\sim}{r}_{G/C'} = \omega \underset{\sim}{k} \times \left(-r\underset{\sim}{e}_n\right) = r\omega \underset{\sim}{e}_t = v\underset{\sim}{e}_t$$

or

$$\boxed{\underset{\sim}{v}_G = r\,\omega\,\underset{\sim}{e}_t = v\,\underset{\sim}{e}_t}$$

The ***acceleration*** of G is calculated by ***differentiating*** $\underset{\sim}{v}_G$.

$$\underset{\sim}{a}_G = \frac{{}^R d}{dt}\left(r\omega\underset{\sim}{e}_t\right) = r\dot{\omega}\underset{\sim}{e}_t + r\omega\dot{\underset{\sim}{e}}_t = r\alpha\underset{\sim}{e}_t + r\omega\left(\dot{\theta}\underset{\sim}{k}\times\underset{\sim}{e}_t\right) = r\alpha\underset{\sim}{e}_t + r\omega\left(\frac{v}{R+r}\underset{\sim}{e}_n\right)$$

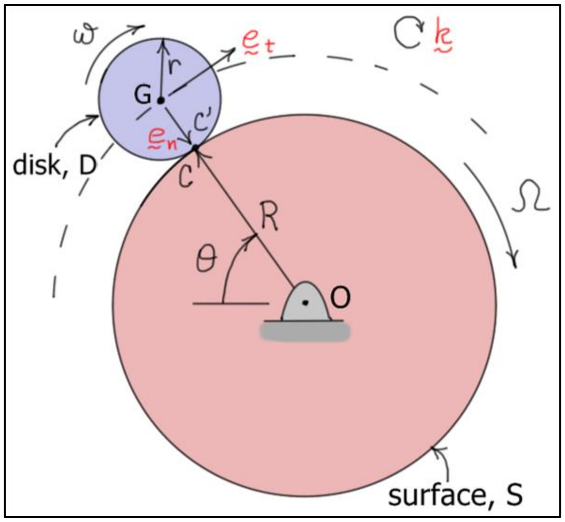
$$= r\alpha\underset{\sim}{e}_t + r\omega\left(\frac{r\omega}{R+r}\underset{\sim}{e}_n\right)$$

or

$$\boxed{\underset{\sim}{a}_G = r\,\alpha\,\underset{\sim}{e}_t + \left(\frac{r^2\omega^2}{R+r}\right)\underset{\sim}{e}_n}$$

Rolling on a Moving Surface

If a rigid body ***rolls*** (without slipping) on a ***moving surface***, the velocities of the ***two contact points***, C and C', must be the same (i.e. $\underset{\sim}{v}_{C'} = \underset{\sim}{v}_C$). For example, consider a ***circular*** disk D that rolls on the rotating ***circular*** surface S as shown in the diagram. As before, the ***velocity*** of the disk center G can be calculated using the ***relative velocity equation***.



$$\underset{\sim}{v}_G = \underset{\sim}{v}_{C'} + \underset{\sim}{v}_{G/C'} = \underset{\sim}{\omega}_S \times \underset{\sim}{r}_{C/O} + \underset{\sim}{\omega}_D \times \underset{\sim}{r}_{G/C'} = \Omega \underset{\sim}{k} \times \left(-R\underset{\sim}{e}_n\right) + \omega \underset{\sim}{k} \times \left(-r\underset{\sim}{e}_n\right) = \left(R\Omega + r\omega\right)\underset{\sim}{e}_t$$

or

$$\boxed{\underset{\sim}{v}_G = \left(R\Omega + r\omega\right)\underset{\sim}{e}_t = v\underset{\sim}{e}_t}$$

Again, the ***acceleration*** of G is found by ***differentiating*** the velocity vector.

$$\underset{\sim}{a}_G = \frac{{}^R d}{dt}\left(R\Omega + r\omega\right)\underset{\sim}{e}_t = \left(R\dot{\Omega} + r\dot{\omega}\right)\underset{\sim}{e}_t + \left(R\Omega + r\omega\right)\dot{\underset{\sim}{e}}_t$$

$$= \left(R\dot{\Omega} + r\dot{\omega}\right)\underset{\sim}{e}_t + \left(R\Omega + r\omega\right)\left(\dot{\theta}\underset{\sim}{k} \times \underset{\sim}{e}_t\right) = \left(R\dot{\Omega} + r\dot{\omega}\right)\underset{\sim}{e}_t + \left(R\Omega + r\omega\right)\left(\frac{v}{R+r}\underset{\sim}{e}_n\right)$$

or

$$\boxed{\underset{\sim}{a}_G = \left(R\dot{\Omega} + r\,\dot{\omega}\right)\underset{\sim}{e}_t + \left(\frac{\left(R\Omega + r\,\omega\right)^2}{R+r}\underset{\sim}{e}_n\right)}$$

Note: Even though the ***velocities*** of the contact points C and C' are ***equal*** (i.e. $\underset{\sim}{v}_{C'} = \underset{\sim}{v}_C$), the ***accelerations*** of these points are ***not equal*** (i.e. $\underset{\sim}{a}_{C'} \neq \underset{\sim}{a}_C$). However, in ***two dimensions***, it can be shown that the ***components*** of these ***accelerations*** along the ***tangent direction*** (indicated by the unit vector $\underset{\sim}{e}_t$) are ***equal*** (i.e. $\underset{\sim}{a}_{C'} \cdot \underset{\sim}{e}_t = \underset{\sim}{a}_C \cdot \underset{\sim}{e}_t$).