

ME 5550 Intermediate Dynamics

Generalized Forces

General Definition

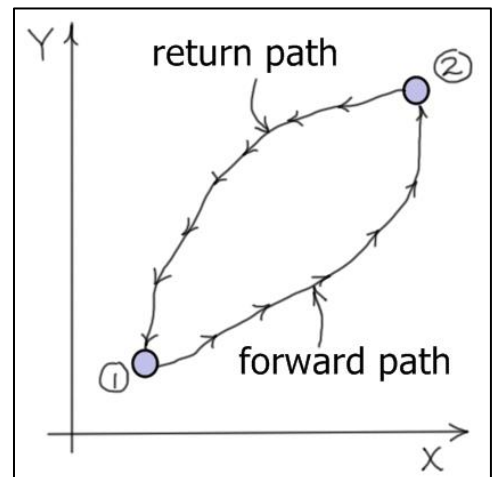
Given a mechanical system whose *configuration* is defined by a set of *generalized coordinates* q_k ($k=1,\dots,n$), the *generalized forces* associated with the “ n ” *generalized coordinates* are defined as follows.

$$F_{q_k} = \sum_{\substack{\text{forces} \\ (i)}} \left(\underline{F}_i \cdot \frac{\partial^R \underline{v}_i}{\partial \dot{q}_k} \right) + \sum_{\substack{\text{torques} \\ (j)}} \left(\underline{M}_j \cdot \frac{\partial^R \underline{\omega}_j}{\partial \dot{q}_k} \right) \quad (k = 1, \dots, n) \quad (1)$$

Here the index “ i ” represents each of the forces, the index “ j ” represents each of the torques, and the index “ k ” represents each of the generalized coordinates. The forces \underline{F}_i and the torques \underline{M}_j that have *non-zero* contributions to this sum are called *active* forces and torques. The *units* of a *generalized force* are generally *pounds* or *foot-pounds* (or, equivalently, Newtons or Newton-meters) depending on the *units* of the *generalized coordinates*.

Conservative and Nonconservative Forces

Consider a particle that moves from position 1 to position 2 along one path (forward path) and back again to position 1 along a second path (return path) as shown in the diagram. A force acting on the particle is said to be *conservative* if the *net work* it does over the closed path is *zero*. Suppose, for example, that the work done by the force as the particle moves from position 1 to position 2 is *positive*, then the force does the same amount of work as the particle returns to position 1, except that this work is *negative*.



In this way, *conservative forces do not permanently add or remove energy from a system*. When the conservative force is doing negative work, the system is said to be gaining *potential energy* that can later be transformed back into *kinetic energy*. It is also true of conservative forces that the work done in moving from one position to another is *independent of the path* of the particle. Examples include weight forces and spring forces and torques.

Forces whose *net work* around a closed circuit are *not zero* are called *nonconservative forces*. The work done by these forces as a particle moves from one position to another is *dependent on the path* of the particle. Examples include *friction* and *damping* forces.

Conservative Forces and Potential Energy (V)

The *generalized force* associated with *conservative forces* and/or *torques* can be written in terms of a *potential energy function*, V . For weight forces and linear spring forces and torques, the potential energy functions are

$$V = m g y \quad (y \text{ is the height of the particle above some arbitrary datum})$$

$$V = \frac{1}{2} k e^2 \quad (e \text{ is the elongation or compression of the spring (units of length)})$$

$$V = \frac{1}{2} k \theta^2 \quad (\theta \text{ is the elongation or compression of the spring (radians or degrees)})$$

For systems with *multiple conservative forces* and *torques*, the *system potential energy* is the *sum* of the potential energies of each of the forces and torques. That is, $V = \sum V_i$.

If conservative forces and/or torques do work on a system (i.e. if they are *active*), their contribution to the *generalized forces* can be calculated using the *general definition* of Eq. (1), *or* they can be calculated using the *potential energy function* V as defined below in Eq. (2).

$$\boxed{F_{q_k} = - \left(\frac{\partial V}{\partial q_k} \right)} \quad (2)$$

Eq. (1) requires the use of *vector expressions* while Eq. (2) uses the *scalar potential energy function*.

Viscous Damping and Rayleigh's Dissipation Function (R)

One type of *nonconservative force* or *torque* is associated with *viscous damping*. One way of modeling this phenomenon is to *assume* that the forces or torques are *proportional* to the *relative velocity* or *relative angular velocity* of the *end-points* of the damping element.

$$\text{Force: } \underline{F} = -c \underline{v} \quad \text{Torque: } \underline{M} = -c \dot{\theta} \underline{n}$$

Here, the direction of the unit vector \underline{n} is defined by the right-hand rule. For these types of nonconservative forces and torques, *Rayleigh's dissipation function* is defined to be

$$\text{Force: } R = \frac{1}{2}c(\dot{x}_2 - \dot{x}_1)^2 \quad \text{Torque: } R = \frac{1}{2}c(\dot{\theta}_2 - \dot{\theta}_1)^2$$

Here, $(\dot{x}_2 - \dot{x}_1)$ represents the **relative velocity** of the end-points of the element and $(\dot{\theta}_2 - \dot{\theta}_1)$ represents the **relative angular velocity** of the end-points element. For systems with **multiple** proportional damping elements, the **system dissipation function** is defined to be the sum of the dissipation functions of the individual dampers. That is, $R = \sum R_i$.

The contribution of these forces and/or torques to the **generalized forces** can be calculated using the **general definition** of Eq. (1), **or** they can be calculated using **Rayleigh's dissipation function** as given below in Eq. (3).

$$F_{q_k} = - \left(\frac{\partial R}{\partial \dot{q}_k} \right) \quad (3)$$

Again, Eq. (1) requires the use of **vector expressions** while Eq. (3) uses the **scalar dissipation function**. For a more detailed presentation of Rayleigh's dissipation function, see the following references.

1. L.A. Pars, *A Treatise on Analytical Dynamics*, Ox Bow Press, 1965.
2. L. Meirovitch, *Methods of Analytical Dynamics*, McGraw-Hill, 1970.
3. L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice-Hall, 1997.
4. H. Baruh, *Analytical Dynamics*, McGraw-Hill, 1999.