ME 6590 Multibody Dynamics
Conversion of Direction Cosines to Euler Parameters
(***Reference***: H. Baruh, *Analytical Dynamics*, McGraw-Hill, 1999)

Given the coordinate transformation matrix $[R]$, the associated ***Euler parameters*** can be ***computed*** as follows. First, recall that $[R]$ can be written in terms of the parameters as follows.

$$[R]=\begin{bmatrix} \left(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2\right) & 2\left(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4\right) & 2\left(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4\right) \\ 2\left(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4\right) & \left(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2\right) & 2\left(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4\right) \\ 2\left(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4\right) & 2\left(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4\right) & \left(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right) \end{bmatrix}$$

Using $R_{ij}\ (i,j=1,2,3)$ to represent the elements of $[R]$, ***observe*** the ***following***.

$$\left.\begin{aligned} R_{11}+R_{22}+R_{33} &= -\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2\right)+3\varepsilon_4^2 \\ &= -\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)+4\varepsilon_4^2 \end{aligned}\right\} \Rightarrow \boxed{R_{11}+R_{22}+R_{33}=4\varepsilon_4^2-1} \tag{1}$$

$$R_{12}-R_{21}=2\left(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4\right)-2\left(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4\right)=4\varepsilon_3\varepsilon_4 \quad \Rightarrow \boxed{R_{12}-R_{21}=4\varepsilon_3\varepsilon_4} \tag{2}$$

$$R_{31}-R_{13}=2\left(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4\right)-2\left(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4\right)=4\varepsilon_2\varepsilon_4 \quad \Rightarrow \boxed{R_{31}-R_{13}=4\varepsilon_2\varepsilon_4} \tag{3}$$

$$R_{23}-R_{32}=2\left(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4\right)-2\left(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4\right)=4\varepsilon_1\varepsilon_4 \quad \Rightarrow \boxed{R_{23}-R_{32}=4\varepsilon_1\varepsilon_4} \tag{4}$$

Using Eqs. (1)-(4), one way of converting the direction cosines (the elements of $[R]$) into Euler parameters is to first calculate ε_4 using Eq. (1).

$$\boxed{\varepsilon_4=\tfrac{1}{2}\sqrt{R_{11}+R_{22}+R_{33}+1}}$$

Then, calculate the other three parameters using Eqs. (2)-(4).

$$\boxed{\varepsilon_1=\frac{R_{23}-R_{32}}{4\varepsilon_4}} \qquad \boxed{\varepsilon_2=\frac{R_{31}-R_{13}}{4\varepsilon_4}} \qquad \boxed{\varepsilon_3=\frac{R_{12}-R_{21}}{4\varepsilon_4}}$$

However, this solution is ***singular*** when $\varepsilon_4=0$ and is ***ill-conditioned*** when ε_4 is ***small***. To omit all singularities, Baruh makes the following observations:

$$4\varepsilon_1^2 = R_{11}-R_{22}-R_{33}+1 \tag{5}$$

$$4\varepsilon_2^2 = -R_{11}+R_{22}-R_{33}+1 \tag{6}$$

$$4\varepsilon_3^2 = -R_{11}-R_{22}+R_{33}+1 \tag{7}$$

$$4\varepsilon_4^2 = R_{11}+R_{22}+R_{33}+1 \tag{8}$$

Using Eqs. (5)-(8), solve for the ***largest*** of ε_1, ε_2, ε_3, and ε_4. Then, using Eqs. (2)-(4) solve for the "other" parameters.

Note: The parameters ***cannot*** be found ***exclusively*** using Eqs. (5)-(8), because the ***algebraic signs*** of the parameters cannot be determined alone from these equations.

<u>Validation of this Approach</u>

As noted above, using Eqs. (5)-(8) gives the absolute values of each of the parameters. Having identified the parameter with the largest absolute value, one must arbitrarily choose the sign of that parameter to solve for the other three. Does it matter which sign is chosen for the largest (in absolute value) parameter?

To answer this question, consider that given a set of parameters $\varepsilon_i \; (i=1,2,3,4)$, the negatives of these same parameters give the same transformation matrix. This occurs because the elements of the matrix all involve products of the parameters. If all four parameters are negated, the same products occur in each element. So, it ***does not matter*** which sign is chosen for the largest parameter as determined by Eqs. (5)-(8). For further illustration of this conclusion, consider the following two examples.

Example 1:

A single rotation of $\theta = 350$ (deg) about the direction $\underset{\sim}{\lambda} = \frac{2}{7}\underset{\sim}{N}_1 - \frac{3}{7}\underset{\sim}{N}_2 + \frac{6}{7}\underset{\sim}{N}_3$ is used to orient a body. Calculate the following: a) $\varepsilon_i \; (i=1,2,3,4)$ and b) $[R]$. Using the result of part (b), calculate c) the Euler parameters using the procedure outlined above and d) the angle and the direction of rotation.

a) $\boxed{\varepsilon_1 = \frac{2}{7}\sin(350/2) \approx 0.024902}$ $\boxed{\varepsilon_2 = -\frac{3}{7}\sin(350/2) \approx -0.037352}$

$\boxed{\varepsilon_3 = \frac{6}{7}\sin(350/2) \approx 0.074705}$ $\boxed{\varepsilon_4 = \cos(350/2) \approx -0.996195}$

b) $$\boxed{[R] \approx \begin{bmatrix} 0.986048 & -0.150702 & -0.070700 \\ 0.146981 & 0.987598 & -0.055195 \\ 0.078141 & 0.044033 & 0.995969 \end{bmatrix}}$$

c) Absolute values of the parameters:

$\boxed{|\varepsilon_1| = \frac{1}{2}\sqrt{R_{11} - R_{22} - R_{33} + 1} \approx 0.024902}$ $\boxed{|\varepsilon_2| = \frac{1}{2}\sqrt{-R_{11} + R_{22} - R_{33} + 1} \approx 0.037352}$

$\boxed{\left|\varepsilon_3\right| = \tfrac{1}{2}\sqrt{-R_{11} - R_{22} + R_{33} + 1} \approx 0.074705}$ $\boxed{\left|\varepsilon_4\right| = \tfrac{1}{2}\sqrt{R_{11} + R_{22} + R_{33} + 1} \approx 0.996195}$

The parameter with the largest absolute value is ε_4, so let $\boxed{\varepsilon_4 = 0.996195}$ and then solve for the other three parameters using Eqs. (2)-(4).

$$\boxed{\varepsilon_1 = \frac{R_{23} - R_{32}}{4\varepsilon_4} \approx -0.024902} \quad \boxed{\varepsilon_2 = \frac{R_{31} - R_{13}}{4\varepsilon_4} \approx 0.037352} \quad \boxed{\varepsilon_3 = \frac{R_{12} - R_{21}}{4\varepsilon_4} \approx -0.074705}$$

Note that these are the negatives of the original parameters.

d) Using the value of ε_4, the angle of rotation can be calculated as follows.

$$\boxed{\theta = 2\cos^{-1}\left(\varepsilon_4\right) = \pm 10 \text{ (deg)}}$$

$\theta = -10 \text{ (deg)}$:

$$\boxed{\lambda_1 = \varepsilon_1 / \sin\left(-10/2\right) \approx 0.285714 \approx \tfrac{2}{7}} \quad \boxed{\lambda_2 = \varepsilon_2 / \sin\left(-10/2\right) \approx -0.428571 \approx -\tfrac{3}{7}}$$

$$\boxed{\lambda_3 = \varepsilon_3 / \sin\left(-10/2\right) \approx 0.857143 \approx \tfrac{6}{7}}$$

$\theta = +10 \text{ (deg)}$:

$$\boxed{\lambda_1 = \varepsilon_1 / \sin\left(10/2\right) \approx -0.285714 \approx -\tfrac{2}{7}} \quad \boxed{\lambda_2 = \varepsilon_2 / \sin\left(10/2\right) \approx 0.428571 \approx \tfrac{3}{7}}$$

$$\boxed{\lambda_3 = \varepsilon_3 / \sin\left(10/2\right) \approx -0.857143 \approx -\tfrac{6}{7}}$$

Note that a $+350 \text{ (deg)}$ rotation about a direction is equivalent to a -10 (deg) rotation about the same direction or a $+10 \text{ (deg)}$ rotation about the negative of that direction.

Example 2:

A single rotation of $\theta = -250 \text{ (deg)}$ about the direction $\underset{\sim}{\lambda} = \tfrac{2}{7}\underset{\sim}{N}_1 - \tfrac{3}{7}\underset{\sim}{N}_2 + \tfrac{6}{7}\underset{\sim}{N}_3$ is used to orient a body. Calculate the following: a) $\varepsilon_i \left(i = 1,2,3,4\right)$ and b) $\left[R\right]$. Using the result of part (b), calculate c) the Euler parameters using the procedure outlined above and d) the angle and the direction of rotation.

a) $\boxed{\varepsilon_1 = \tfrac{2}{7}\sin\left(-250/2\right) \approx -0.234043}$ $\boxed{\varepsilon_2 = -\tfrac{3}{7}\sin\left(-250/2\right) \approx 0.351065}$

$\boxed{\varepsilon_3 = \tfrac{6}{7}\sin\left(-250/2\right) \approx -0.702130}$ $\boxed{\varepsilon_4 = \cos\left(-250/2\right) \approx -0.573576}$

b) $$[R] \approx \begin{bmatrix} -0.232467 & 0.641122 & 0.731383 \\ -0.969780 & -0.095527 & -0.224503 \\ -0.074067 & -0.761471 & 0.643954 \end{bmatrix}$$

c) Absolute values of the parameters:

$$|\varepsilon_1| = \tfrac{1}{2}\sqrt{R_{11} - R_{22} - R_{33} + 1} \approx 0.234044 \qquad |\varepsilon_2| = \tfrac{1}{2}\sqrt{-R_{11} + R_{22} - R_{33} + 1} \approx 0.351065$$

$$|\varepsilon_3| = \tfrac{1}{2}\sqrt{-R_{11} - R_{22} + R_{33} + 1} \approx 0.702130 \qquad |\varepsilon_4| = \tfrac{1}{2}\sqrt{R_{11} + R_{22} + R_{33} + 1} \approx 0.573576$$

The parameter with the largest absolute value is ε_3, so let $\boxed{\varepsilon_3 = 0.702130}$ and then solve for the other three parameters using Eqs. (2)-(4).

$$\varepsilon_4 = \frac{R_{12} - R_{21}}{4\varepsilon_3} \approx 0.573576 \qquad \varepsilon_1 = \frac{R_{23} - R_{32}}{4\varepsilon_4} \approx 0.234043 \qquad \varepsilon_2 = \frac{R_{31} - R_{13}}{4\varepsilon_4} \approx -0.351065$$

Note again that these are the negatives of the original parameters.

d) Using the value of ε_4, the angle of rotation can be calculated as follows.

$$\theta = 2\cos^{-1}(\varepsilon_4) = \pm 110 \text{ (deg)}$$

$\theta = +110$ (deg):

$$\lambda_1 = \varepsilon_1 / \sin(110/2) \approx 0.285714 \approx \tfrac{2}{7} \qquad \lambda_2 = \varepsilon_2 / \sin(110/2) \approx -0.428571 \approx -\tfrac{3}{7}$$

$$\lambda_3 = \varepsilon_3 / \sin(110/2) \approx 0.857143 \approx \tfrac{6}{7}$$

$\theta = -110$ (deg):

$$\lambda_1 = \varepsilon_1 / \sin(-110/2) \approx -0.285714 \approx -\tfrac{2}{7} \qquad \lambda_2 = \varepsilon_2 / \sin(-110/2) \approx 0.428571 \approx \tfrac{3}{7}$$

$$\lambda_3 = \varepsilon_3 / \sin(-110/2) \approx -0.857143 \approx -\tfrac{6}{7}$$

Note that a -250 (deg) rotation about a direction is equivalent to a $+110$ (deg) rotation about the same direction or a -110 (deg) rotation about the negative of that direction.