An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 6

Rigid Body Orientation, Orientation Parameters, and Angular Velocity

Summary

In Unit 5 the concept of using ***orientation angles*** to describe the orientation and angular velocity of a rigid body was presented. It was noted, however, that ***all orientation angle sequences*** display a ***singularity*** at some orientation and that this can cause problems for computer programs that use them. This unit shows how to use the ***orientation parameters*** known as ***Euler parameters*** to remedy this situation.

Some ***limited proofs*** of properties associated with Euler parameters are provided in the ***Addendum*** to this unit. A detailed understanding of those proofs, however, is ***not necessary*** when applying these concepts.

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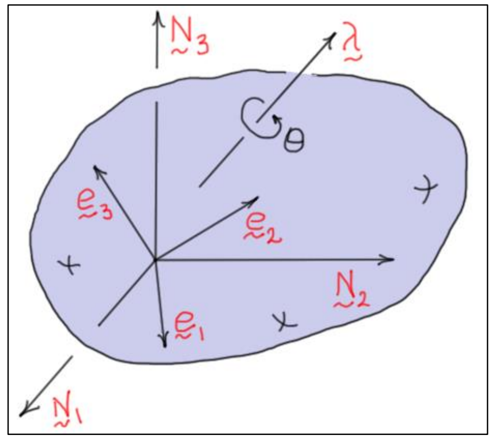
Orientation of a Rigid Body Using Euler Parameters

Euler's Theorem on Rotation

Consider the ***rigid body*** shown in the figure. Let $A:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ be a set of unit vectors representing the ***base reference frame*** and $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ be a set that represents the ***body-fixed reference frame***, and assume that the two frames are ***initially*** aligned. Then, ***Euler's Theorem on Rotation*** states that the rigid body $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ can be moved into ***any arbitrary orientation*** relative to the base frame by a rotation about a single axis. In the diagram, θ represents the ***angle*** of rotation, and the ***unit vector*** $\underset{\sim}{\lambda}$ represents the ***direction*** of rotation (in the right-hand sense).



Euler Parameters

The unit vector $\underset{\sim}{\lambda}$ and the angle θ can be related to a set of ***four orientation parameters*** called the ***Euler parameters***. First, because the rotation occurs about $\underset{\sim}{\lambda}$, the unit vector has the ***same components*** in the base frame and in the body-fixed frame. That is,

$$\boxed{\underset{\sim}{\lambda}=\lambda_1\underset{\sim}{N}_1+\lambda_2\underset{\sim}{N}_2+\lambda_3\underset{\sim}{N}_3=\lambda_1\,\underset{\sim}{e}_1+\lambda_2\,\underset{\sim}{e}_2+\lambda_3\,\underset{\sim}{e}_3}$$

The ***four Euler parameters*** are defined in terms of these components and the angle of rotation as follows.

$$\boxed{\varepsilon_i=\lambda_i\sin(\theta/2)}\quad(i=1,2,3)\qquad\qquad\boxed{\varepsilon_4=\cos(\theta/2)}$$

Properties of the Euler Parameters

The following is a list of ***useful properties*** associated with Euler parameters. It can be shown that:

1. The Euler parameters are ***not independent***, because $\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2=1$.
2. The transformation matrix $\left[R\right]$ that relates the base-fixed and body-fixed unit vectors can be written in terms of the Euler parameters as follows:

$$\boxed{\begin{Bmatrix}\underset{\sim}{e}_1\\\underset{\sim}{e}_2\\\underset{\sim}{e}_3\end{Bmatrix}=\left[R\right]\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}=\begin{bmatrix}(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)\\ 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & (-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)\\ 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) & (-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)\end{bmatrix}\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}}$$

3. The transformation matrix $\left[R\right]$ is ***orthogonal***, so $\left[R\right]^{-1}=\left[R\right]^T$.
4. The ***base-frame components*** $\left(\omega_i\ (i=1,2,3)\right)$ of the angular velocity of the body relative to the base frame can be written in terms of the ***Euler parameters*** using matrix notion as follows.

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix} = 2\left[E\right]\begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} \qquad \text{or} \qquad \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} = \tfrac{1}{2}\left[E\right]^T \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix}$$

Here, $\left[E\right]$ is the ***orthogonal matrix***

$$\left[E\right] = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & -\varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

5. The ***body-frame components*** $\left(\omega_i' \; (i=1,2,3)\right)$ of the angular velocity of the body relative to the base frame can be written in terms of the ***Euler parameters*** using matrix notion as follows.

$$\begin{Bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \\ 0 \end{Bmatrix} = 2\left[E'\right]\begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} \qquad \text{or} \qquad \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} = \tfrac{1}{2}\left[E'\right]^T \begin{Bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \\ 0 \end{Bmatrix}$$

Here, $\left[E'\right]$ is the ***orthogonal matrix***

$$\left[E'\right] = \begin{bmatrix} \varepsilon_4 & \varepsilon_3 & -\varepsilon_2 & -\varepsilon_1 \\ -\varepsilon_3 & \varepsilon_4 & \varepsilon_1 & -\varepsilon_2 \\ \varepsilon_2 & -\varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

The results listed in property (5) are ***analogous*** to those developed in Unit 5 that relate the ***angular velocity components*** to the derivatives of a set of ***orientation angles***. Note that ***no singularities*** exist in the kinematic equations shown here, so many computer programs use Euler parameters to ***avoid computational singularities***. However, they may ***communicate*** with the "user" using orientation angles which are easier to visualize.

The development of some of the properties of Euler parameters listed above requires a much more involved analysis than that required for orientation angles. For that reason, the results are presented ***initially without proof***. Some ***limited proofs*** of these properties are given in the ***Addendum*** to this unit. A detailed understanding of those proofs, however, is not necessary to use the concepts.

Since the orientation of a body can be described using either orientation angles or Euler parameters, for any given set of orientation angles (or Euler parameters), an equivalent set of Euler parameters (or orientation angles) can be found. The following two sections discuss this process.

Conversion of Orientation Angles to Euler Parameters

(***Reference***: H. Baruh, *Analytical Dynamics*, McGraw-Hill, 1999)

Given a set of orientation angles, the transformation matrix $\left[R\right]$ is easily calculated using the methods discussed in Unit 5. An equivalent set of ***Euler parameters*** can be ***computed*** from $\left[R\right]$ as follows. First, recall that $\left[R\right]$ can be written as

$$\left[R\right]=\begin{bmatrix} (\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & (-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) & (-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2) \end{bmatrix}$$

The following ***four observations*** can be made from this result.

$$R_{11}-R_{22}-R_{33}=(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)-(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)-(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)=3\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2-\varepsilon_4^2$$
$$=4\varepsilon_1^2-\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)$$
$$\Rightarrow \quad \boxed{4\varepsilon_1^2=R_{11}-R_{22}-R_{33}+1}$$

$$-R_{11}+R_{22}-R_{33}=-(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)+(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)-(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)=-\varepsilon_1^2+3\varepsilon_2^2-\varepsilon_3^2-\varepsilon_4^2$$
$$=4\varepsilon_2^2-\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)$$
$$\Rightarrow \quad \boxed{4\varepsilon_2^2=-R_{11}+R_{22}-R_{33}+1}$$

$$-R_{11}-R_{22}+R_{33}=-(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)-(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)+(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)=-\varepsilon_1^2-\varepsilon_2^2+3\varepsilon_3^2-\varepsilon_4^2$$
$$=4\varepsilon_3^2-\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)$$
$$\Rightarrow \quad \boxed{4\varepsilon_3^2=-R_{11}-R_{22}+R_{33}+1}$$

$$R_{11}+R_{22}+R_{33}=(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)+(-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2)+(-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)=-\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+3\varepsilon_4^2$$
$$=-\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)+4\varepsilon_4^2$$
$$\Rightarrow \quad \boxed{4\varepsilon_4^2=R_{11}+R_{22}+R_{33}+1}$$

Note that these equations ***cannot be used alone*** to find the Euler parameters, because each equation involves the ***square*** of the parameter. Hence the ***algebraic sign*** of the parameter is not determined. However, to avoid computational problems when one of the Euler parameters is ***small*** or ***zero***, it is recommended that these four equations be used to identify the Euler parameter having the ***largest absolute value***. Then using this value, compute the other Euler parameters using one of the following equations. Note that each of these equations will determine the ***necessary signs*** of the remaining parameters. A ***MATLAB script*** to perform this conversion for a 1-2-3 body-fixed orientation angle sequence is provided in Unit 10.

$$R_{12}+R_{21}=4\varepsilon_1\varepsilon_2 \quad R_{12}-R_{21}=4\varepsilon_3\varepsilon_4 \quad R_{31}+R_{13}=4\varepsilon_1\varepsilon_3 \quad R_{31}-R_{13}=4\varepsilon_2\varepsilon_4$$

$$R_{23}+R_{32}=4\varepsilon_2\varepsilon_3 \quad R_{23}-R_{32}=4\varepsilon_1\varepsilon_4$$

Validation of this Approach

Having identified the Euler parameter with the ***largest absolute value***, one must arbitrarily choose the sign of that parameter to solve for the other three. Does it matter which sign is chosen for the largest parameter?

To answer this question, consider that given a set of parameters $\varepsilon_i \; (i=1,2,3,4)$, the ***negatives*** of these same parameters yield the same transformation matrix. This occurs because all the elements of the transformation matrix involve products of the parameters. If all four parameters are negated, the same products occur in each element. So, it ***does not matter*** which sign is chosen for the largest parameter. For further illustration of this conclusion, consider the first two of the following examples.

Example 1:

A single rotation of $\theta = 350$ (deg) about the direction $\underset{\sim}{\lambda} = \frac{2}{7}\underset{\sim}{N}_1 - \frac{3}{7}\underset{\sim}{N}_2 + \frac{6}{7}\underset{\sim}{N}_3$ is used to orient a body.

Find:

a) $\varepsilon_i \; (i=1,2,3,4)$ and $[R]$ associated with this rotation

b) Using $[R]$ of part (a), calculate the Euler parameters using the procedure outlined above.

c) θ and $\underset{\sim}{\lambda}$ associated with the results of part (b)

a) $$\varepsilon_1 = \tfrac{2}{7}\sin(350/2) \approx 0.024902 \quad \varepsilon_2 = -\tfrac{3}{7}\sin(350/2) \approx -0.037352$$

$$\varepsilon_3 = \tfrac{6}{7}\sin(350/2) \approx 0.074705 \quad \varepsilon_4 = \cos(350/2) \approx -0.996195$$

$$[R] \approx \begin{bmatrix} 0.986048 & -0.150702 & -0.070700 \\ 0.146981 & 0.987598 & -0.055195 \\ 0.078141 & 0.044033 & 0.995969 \end{bmatrix}$$

b) Absolute values of the parameters:

$$|\varepsilon_1| = \tfrac{1}{2}\sqrt{R_{11}-R_{22}-R_{33}+1} \approx 0.024902 \quad |\varepsilon_2| = \tfrac{1}{2}\sqrt{-R_{11}+R_{22}-R_{33}+1} \approx 0.037352$$

$$|\varepsilon_3| = \tfrac{1}{2}\sqrt{-R_{11}-R_{22}+R_{33}+1} \approx 0.074705 \quad |\varepsilon_4| = \tfrac{1}{2}\sqrt{R_{11}+R_{22}+R_{33}+1} \approx 0.996195$$

The parameter with the largest absolute value is ε_4, so let $\varepsilon_4 = 0.996195$ and then solve for the other three parameters.

$$\varepsilon_1 = \frac{R_{23}-R_{32}}{4\varepsilon_4} \approx -0.024902 \quad \varepsilon_2 = \frac{R_{31}-R_{13}}{4\varepsilon_4} \approx 0.037352 \quad \varepsilon_3 = \frac{R_{12}-R_{21}}{4\varepsilon_4} \approx -0.074705$$

Note that these values are the ***negatives*** of the original parameters.

d) Using the value of ε_4, the angle of rotation can be calculated as follows.

$$\boxed{\theta = 2\cos^{-1}\left(\varepsilon_4\right) = \pm 10 \text{ (deg)}}$$

$\theta = -10$ (deg) :

$$\boxed{\lambda_1 = \varepsilon_1 / \sin\left(-10/2\right) \approx 0.285714 \approx \tfrac{2}{7}} \quad \boxed{\lambda_2 = \varepsilon_2 / \sin\left(-10/2\right) \approx -0.428571 \approx -\tfrac{3}{7}}$$

$$\boxed{\lambda_3 = \varepsilon_3 / \sin\left(-10/2\right) \approx 0.857143 \approx \tfrac{6}{7}}$$

$\theta = +10$ (deg) :

$$\boxed{\lambda_1 = \varepsilon_1 / \sin\left(10/2\right) \approx -0.285714 \approx -\tfrac{2}{7}} \quad \boxed{\lambda_2 = \varepsilon_2 / \sin\left(10/2\right) \approx 0.428571 \approx \tfrac{3}{7}}$$

$$\boxed{\lambda_3 = \varepsilon_3 / \sin\left(10/2\right) \approx -0.857143 \approx -\tfrac{6}{7}}$$

Note that a $+350$ (deg) rotation about a direction is ***equivalent*** to a -10 (deg) rotation about the ***same direction*** or a $+10$ (deg) rotation about the ***negative of that direction***.

Example 2:

A single rotation of $\theta = -250$ (deg) about the direction $\underset{\sim}{\lambda} = \frac{2}{7}\underset{\sim}{N}_1 - \frac{3}{7}\underset{\sim}{N}_2 + \frac{6}{7}\underset{\sim}{N}_3$ is used to orient a body.

Find:

a) $\varepsilon_i \left(i = 1,2,3,4\right)$ and $\left[R\right]$ corresponding to this rotation

b) Using $\left[R\right]$ of part (a), calculate the Euler parameters using the procedure outlined above.

c) θ and $\underset{\sim}{\lambda}$ associated with the results of part (b)

a) $\boxed{\varepsilon_1 = \tfrac{2}{7}\sin\left(-250/2\right) \approx -0.234043}$ $\boxed{\varepsilon_2 = -\tfrac{3}{7}\sin\left(-250/2\right) \approx 0.351065}$

$$\boxed{\varepsilon_3 = \tfrac{6}{7}\sin\left(-250/2\right) \approx -0.702130} \quad \boxed{\varepsilon_4 = \cos\left(-250/2\right) \approx -0.573576}$$

$$\boxed{\left[R\right] \approx \begin{bmatrix} -0.232467 & 0.641122 & 0.731383 \\ -0.969780 & -0.095527 & -0.224503 \\ -0.074067 & -0.761471 & 0.643954 \end{bmatrix}}$$

c) Absolute values of the parameters:

$$\boxed{\left|\varepsilon_1\right| = \tfrac{1}{2}\sqrt{R_{11} - R_{22} - R_{33} + 1} \approx 0.234044} \quad \boxed{\left|\varepsilon_2\right| = \tfrac{1}{2}\sqrt{-R_{11} + R_{22} - R_{33} + 1} \approx 0.351065}$$

$$\boxed{\left|\varepsilon_3\right| = \tfrac{1}{2}\sqrt{-R_{11} - R_{22} + R_{33} + 1} \approx 0.702130} \quad \boxed{\left|\varepsilon_4\right| = \tfrac{1}{2}\sqrt{R_{11} + R_{22} + R_{33} + 1} \approx 0.573576}$$

The parameter with the largest absolute value is ε_3, so let $\boxed{\varepsilon_3 = 0.702130}$ and then solve for the other three parameters.

$$\varepsilon_4 = \frac{R_{12} - R_{21}}{4\varepsilon_3} \approx 0.573576 \quad \varepsilon_1 = \frac{R_{23} - R_{32}}{4\varepsilon_4} \approx 0.234043 \quad \varepsilon_2 = \frac{R_{31} - R_{13}}{4\varepsilon_4} \approx -0.351065$$

Note again that these values are the ***negatives*** of the original parameters.

d) Using the value of ε_4, the angle of rotation can be calculated as follows.

$$\theta = 2\cos^{-1}(\varepsilon_4) = \pm 110 \text{ (deg)}$$

$\theta = +110$ (deg) :

$$\lambda_1 = \varepsilon_1 / \sin(110/2) \approx 0.285714 \approx \tfrac{2}{7} \quad \lambda_2 = \varepsilon_2 / \sin(110/2) \approx -0.428571 \approx -\tfrac{3}{7}$$

$$\lambda_3 = \varepsilon_3 / \sin(110/2) \approx 0.857143 \approx \tfrac{6}{7}$$

$\theta = -110$ (deg) :

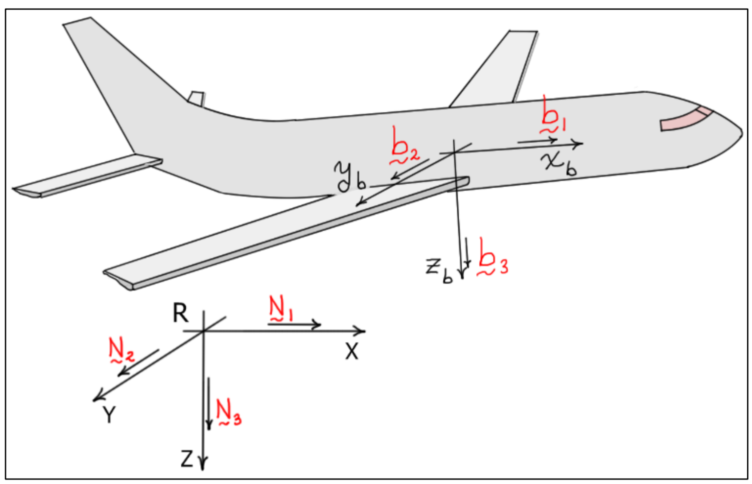
$$\lambda_1 = \varepsilon_1 / \sin(-110/2) \approx -0.285714 \approx -\tfrac{2}{7} \quad \lambda_2 = \varepsilon_2 / \sin(-110/2) \approx 0.428571 \approx \tfrac{3}{7}$$

$$\lambda_3 = \varepsilon_3 / \sin(-110/2) \approx -0.857143 \approx -\tfrac{6}{7}$$

Note that a -250 (deg) rotation about a direction is ***equivalent*** to a $+110$ (deg) rotation about the ***same direction*** or a -110 (deg) rotation about the ***negative of that direction***.

Example 3: Conversion of Orientation Angles to Euler Parameters

The orientation of an aircraft can be defined by a 3-2-1 body-fixed rotation sequence. As before, the ***body axes*** (x_b, y_b, z_b) are initially aligned with the ***fixed frame axes*** (X, Y, Z). It is common to refer to the three angles as ψ, θ, and ϕ. The following values are given for the three angles:



$\psi = 135$ (deg) , $\theta = 15$ (deg) , $\phi = 25$ (deg)

Find:

ε_i $(i = 1, 2, 3, 4)$ associated with this orientation

Solution:

First, using the transformation matrix derived in Unit 5 for a 3-2-1 orientation angle sequence, calculate the transformation matrix $[R]$.

$$[R]=\begin{bmatrix} C_\psi C_\theta & S_\psi C_\theta & -S_\theta \\ C_\psi S_\theta S_\phi - S_\psi C_\phi & S_\psi S_\theta S_\phi + C_\psi C_\phi & C_\theta S_\phi \\ C_\psi S_\theta C_\phi + S_\psi S_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi & C_\theta C_\phi \end{bmatrix} = \begin{bmatrix} -0.683013 & 0.683013 & -0.258819 \\ -0.718201 & -0.563512 & 0.408218 \\ 0.132970 & 0.464702 & 0.875426 \end{bmatrix}$$

Then, calculate the squares of the Euler parameters using the elements of $[R]$.

$$\varepsilon_1^2 = \left(R_{11} - R_{22} - R_{33} + 1\right)/4 = \left(-0.683013 + 0.563512 - 0.875426 + 1\right)/4 = 0.001268$$

$$\varepsilon_2^2 = \left(-R_{11} + R_{22} - R_{33} + 1\right)/4 = \left(0.683013 - 0.563512 - 0.875426 + 1\right)/4 = 0.061019$$

$$\varepsilon_3^2 = \left(-R_{11} - R_{22} + R_{33} + 1\right)/4 = \left(0.683013 + 0.563512 + 0.875426 + 1\right)/4 = \boxed{0.780488} \Leftarrow \text{largest value}$$

$$\varepsilon_4^2 = \left(R_{11} + R_{22} + R_{33} + 1\right)/4 = \left(-0.683013 - 0.563512 + 0.875426 + 1\right)/4 = 0.157225$$

Noting ε_3^2 has the ***largest value***, proceed as follows. First, calculate ε_3 as the ***positive square root*** of ε_3^2, and then compute the other parameters as follows

$$\boxed{\varepsilon_3 = \sqrt{0.780488} = 0.883452}$$

$$\boxed{\varepsilon_1 = \frac{\left(R_{31} + R_{13}\right)}{4\varepsilon_3} = \frac{\left(0.132970 - 0.258819\right)}{4(0.883452)} = -0.035613}$$

$$\boxed{\varepsilon_2 = \frac{\left(R_{23} + R_{32}\right)}{4\varepsilon_3} = \frac{\left(0.408218 + 0.464702\right)}{4(0.883452)} = 0.247020}$$

$$\boxed{\varepsilon_4 = \frac{\left(R_{12} - R_{21}\right)}{4\varepsilon_3} = \frac{\left(0.683013 + 0.718201\right)}{4(0.883452)} = 0.396517}$$

These results can be checked by simply substituting the parameter values into the transformation matrix $[R]$ and comparing the resulting matrix to the one associated with the orientation angles.

$$[R]=\begin{bmatrix} (\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & (-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) & (-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \end{bmatrix} = \begin{bmatrix} -0.683012 & 0.683013 & -0.258820 \\ -0.718202 & -0.563511 & 0.408218 \\ 0.132971 & 0.464703 & 0.875426 \end{bmatrix}$$

This result ***agrees*** with the original matrix to 5 ***significant figures***.

Conversion of Euler Parameters to Orientation Angles

A similar process can be used to calculate a set of orientation angles that are equivalent to a set of Euler parameters. First, the Euler parameters are used to compute the transformation matrix $[R]$. Then advantage is taken of the ***form*** of the transformation matrix associated with the orientation angles of interest to calculate the angles. Since the form of the transformation matrix is unique to the orientation angle sequence, the algorithm used to compute the angles will be ***different*** for each sequence. The following example reverses the process of Example 3 to find the orientation angles.

Example 4:

Given the values of the Euler parameters:

$$\varepsilon_1 = -0.035613,\ \varepsilon_2 = 0.247020,\ \varepsilon_3 = 0.883452,\ \varepsilon_4 = 0.396517$$

Find:

A set of 3-2-1 body-fixed orientation angles associated with this orientation

Solution:

The transformation matrix associated with these Euler parameter values was found in Example 3 to be

$$[R] = \begin{bmatrix} (\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & (-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) & (-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) \end{bmatrix} = \begin{bmatrix} -0.683012 & 0.683013 & -0.258820 \\ -0.718202 & -0.563511 & 0.408218 \\ 0.132971 & 0.464703 & 0.875426 \end{bmatrix}$$

To calculate a set of orientation angles associated with this transformation matrix, advantage is taken of the ***form*** of the transformation matrix for those angles. It was noted in Example 3 that the form of the transformation matrix for a set of 3-2-1 body-fixed orientation angles is

$$[R] = \begin{bmatrix} C_\psi C_\theta & S_\psi C_\theta & -S_\theta \\ C_\psi S_\theta S_\phi - S_\psi C_\phi & S_\psi S_\theta S_\phi + C_\psi C_\phi & C_\theta S_\phi \\ C_\psi S_\theta C_\phi + S_\psi S_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi & C_\theta C_\phi \end{bmatrix}$$

Using this result, the angles can be calculated using the following ***procedure***.

1. To find the value of θ, use element $(1,3)$ of $[R]$.

   $$\boxed{\theta = \sin^{-1}(-R_{13}) = \sin^{-1}\left(\sin(\theta)\right) = \sin^{-1}(0.258820) = \begin{cases} 15\ (\text{deg}) \\ 180 - 15 = 165\ (\text{deg}) \end{cases}}$$

   Either value can be chosen.

2. To find the value of ψ, use the $(1,1)$ and $(1,2)$ elements of $[R]$.

   $$\boxed{\psi = \tan^{-1}\left(\frac{R_{12}}{R_{11}}\right) = \tan^{-1}\left(\frac{\sin(\psi)\cos(\theta)}{\cos(\psi)\cos(\theta)}\right) = \tan^{-1}\left(\frac{0.683013}{-0.683012}\right) = \begin{cases} 135\ (\text{deg}) \\ -45\ (\text{deg}) \end{cases}} \qquad (\cos(\theta) \neq 0)$$

   If θ is chosen to be $15\ (\text{deg})$, then $\cos(\theta) > 0$, and the $(1,2)$ and $(1,1)$ elements of $[R]$ indicate that $\sin(\psi) > 0$ and $\cos(\psi) < 0$. In this case, the proper choice is $\psi = 135\ (\text{deg})$. If θ is chosen to be $165\ (\text{deg})$, then $\cos(\theta) < 0$, and the $(1,2)$ and $(1,1)$ elements of $[R]$ indicate that $\sin(\psi) < 0$ and $\cos(\psi) > 0$. In this case, the proper choice is $\psi = -45\ (\text{deg})$.

3. To find the value of ϕ, use the $(2,3)$ and $(3,3)$ elements of $[R]$.

$$\phi = \tan^{-1}\left(\frac{R_{23}}{R_{33}}\right) = \tan^{-1}\left(\frac{\cos(\theta)\sin(\phi)}{\cos(\theta)\cos(\phi)}\right) = \tan^{-1}\left(\frac{0.408218}{0.875426}\right) = \begin{cases} 25 \text{ (deg)} \\ 205 \text{ (deg)} \end{cases} \qquad (\cos(\theta) \neq 0)$$

If θ is chosen to be 15 (deg), then $\cos(\theta) > 0$, and the $(2,3)$ and $(3,3)$ elements of $[R]$ indicate that $\sin(\phi) > 0$ and $\cos(\phi) > 0$. In this case, the proper choice is $\phi = 25$ (deg). If θ is chosen to be 165 (deg), then $\cos(\theta) < 0$, and the $(2,3)$ and $(3,3)$ elements of $[R]$ indicate that $\sin(\phi) < 0$ and $\cos(\phi) < 0$. In this case, the proper choice is $\phi = 205$ (deg).

Notes:

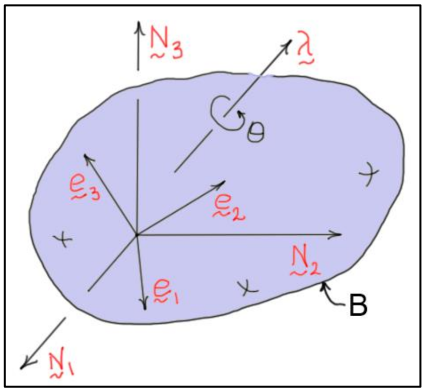
1. The 3-2-1 angle sequences $(135, 15, 25)$ (deg) and $(-45, 165, 205)$ (deg) are ***equivalent*** sequences in that they provide the ***same transformation matrix*** and the ***same final position***.
2. The above procedure ***fails*** when $\cos(\theta) = 0$, that is, when $\theta = \pi/2$ (rad) $= 90$ (deg). Step 1 ***works***, but steps 2 and 3 ***fail*** because both the numerators and denominators of the inverse tangent functions are ***zero***.
3. When $\theta = 90$ (deg), the ψ and ϕ rotations occur about the ***same axis***, and consequently ***cannot be individually determined***. Only their ***sum*** can be found. One approach in this case is simply to ***set*** one of the angles to ***zero*** and ***solve*** for the ***other***. So, for example, if $\theta = 90$ (deg) and ϕ is ***set*** to ***zero***, ψ can be found using the $(3,1)$ entry in the transformation matrix.

$$\psi = \cos^{-1}(R_{31}) = \cos^{-1}\left(C_\psi \underbrace{S_\theta}_{=1} C_\phi + S_\psi \underbrace{S_\phi}_{=0}\right) = \cos^{-1}(C_\psi)$$

Similar approaches can be taken for ***other*** orientation angle sequences.

Exercises:

6.1 A rigid body B with unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is oriented relative to a base reference frame with unit vectors $(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$ by rotating the body by a single angle $\theta = 60$ (deg) about a direction indicated by the unit vector $\underset{\sim}{\lambda} = \frac{2}{7}\underset{\sim}{N}_1 - \frac{3}{7}\underset{\sim}{N}_2 + \frac{6}{7}\underset{\sim}{N}_3$. Assuming the unit vectors of the body are initially aligned with those of the base frame, complete the following.



a) Find the four Euler parameters associated with this orientation.

b) Find the transformation matrix $[R]$ that can be used to express the body-fixed unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ in terms of the base unit vectors $(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$.

c) Using the transformation matrix $[R]$, express each of the unit vectors $\underset{\sim}{e}_i$ $(i = 1, 2, 3)$ in terms of the unit vectors $\underset{\sim}{N}_i$ $(i = 1, 2, 3)$.

Answers:

$\varepsilon_1 = 0.142857;\ \varepsilon_2 = -0.214286;\ \varepsilon_3 = 0.428571;\ \varepsilon_4 = 0.866025$

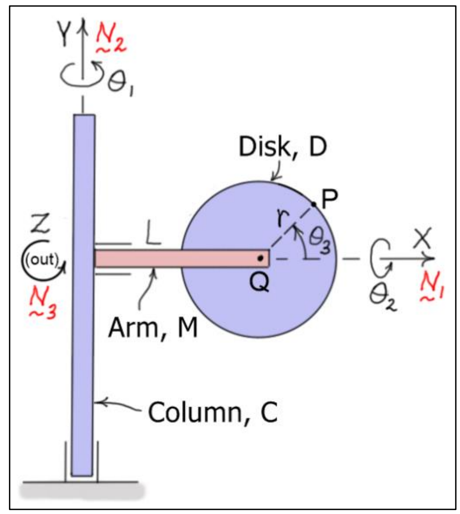
$$[R] = \begin{bmatrix} 0.540816 & 0.681083 & 0.493603 \\ -0.803532 & 0.591837 & 0.063762 \\ -0.248705 & -0.431109 & 0.867347 \end{bmatrix}$$

$\underset{\sim}{e}_1 = 0.540816\,\underset{\sim}{N}_1 + 0.681083\,\underset{\sim}{N}_2 + 0.493603\,\underset{\sim}{N}_3$

$\underset{\sim}{e}_2 = -0.803532\,\underset{\sim}{N}_1 + 0.591837\,\underset{\sim}{N}_2 + 0.063762\,\underset{\sim}{N}_3$

$\underset{\sim}{e}_3 = -0.248705\,\underset{\sim}{N}_1 - 0.431109\,\underset{\sim}{N}_2 + 0.867347\,\underset{\sim}{N}_3$

6.2 The disk D is oriented relative to the ground frame $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$ using a 2-1-3 body-fixed rotation sequence as indicated in the diagram. Given the angles $\theta_1 = -45\,(\text{deg})$, $\theta_2 = 30\ (\text{deg})$, and $\theta_3 = 60\ (\text{deg})$, and their time-derivatives $\dot{\theta}_1 = 0.5\ (\text{rad/s})$, $\dot{\theta}_2 = -1.5\ (\text{rad/s})$, and $\dot{\theta}_3 = 2\ (\text{rad/s})$, complete the following.



a) Find the Euler parameters associated with this orientation.

b) Find the body-fixed angular velocity components associated with the rate of change of this orientation.

c) Find the time-derivatives of the Euler parameters associated with the rate of change of this orientation.

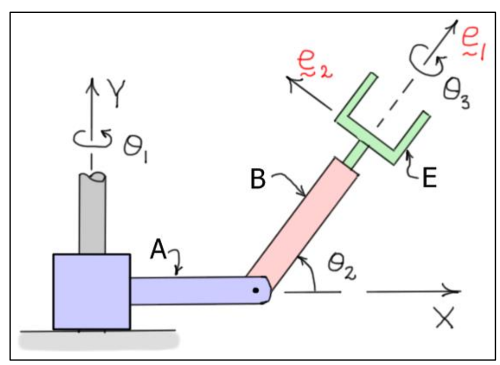
Answers:

$\varepsilon_1 = 0.022260;\ \varepsilon_2 = -0.439680;\ \varepsilon_3 = 0.531976;\ \varepsilon_4 = 0.723317$

$\omega_1' = -0.375\ (\text{rad/s});\quad \omega_2' = 1.51554\ (\text{rad/s});\quad \omega_3' = 1.75\ (\text{rad/s})$

$\dot{\varepsilon}_1 = 0.652214;\ \dot{\varepsilon}_2 = 0.667333;\ \dot{\varepsilon}_3 = 698475;\ \dot{\varepsilon}_4 = -0.128128$

6.3 The system shown consists of three components, the arms A and B and the end-effector E. The orientation of E relative to a fixed frame is described by the three angles shown. Note that the sequence of rotations θ_1, θ_2, and θ_3 is a 2-3-1 body-fixed rotation sequence. The angles associated with a specific orientation of E are given as:



$$\theta_1 = 30\ (\text{deg}) \qquad \theta_2 = 60\ (\text{deg}) \qquad \theta_3 = 40\ (\text{deg})$$

a) Find the transformation matrix relating the unit vectors fixed in E to those of the base frame.

b) Find the Euler parameters associated with that position.

c) Develop a method for finding the 2-3-1 angle sequence given only the transformation matrix.

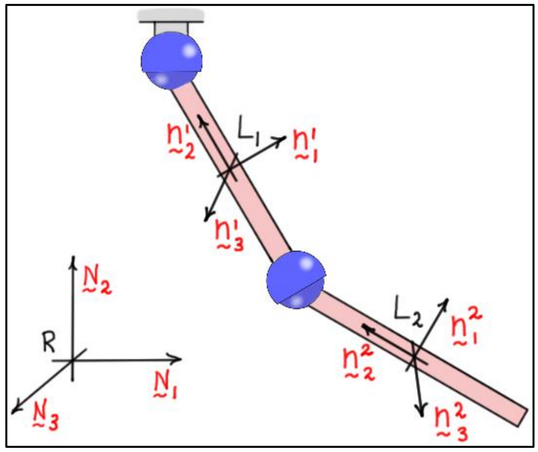
Answers:

$$[R]=\begin{bmatrix} C_1C_2 & S_2 & -S_1C_2 \\ S_1S_3-C_1S_2C_3 & C_2C_3 & C_1S_3+S_1S_2C_3 \\ S_1C_3+C_1S_2S_3 & -C_2S_3 & C_1C_3-S_1S_2S_3 \end{bmatrix}=\begin{bmatrix} 0.433013 & 0.866025 & -0.250000 \\ -0.253140 & 0.383022 & 0.888377 \\ 0.865113 & -0.321394 & 0.385079 \end{bmatrix}$$

$\varepsilon_1=0.407711$ $\varepsilon_2=0.375809$ $\varepsilon_3=0.377175$ $\varepsilon_4=0.741808$

6.4 The system shown is a three-dimensional double pendulum (or arm). The first link is connected to ground and the second link is connected to the first with ball-and-socket joints. The orientation of each link is defined relative to the base-frame R using a 3-1-3 body-fixed rotation sequence. The angles for the second link are:

$\theta_1=30$ (deg) $\theta_2=60$ (deg) $\theta_3=20$ (deg)



a) Find the transformation matrix relating the system of unit vectors fixed in L_2 to those of the base system.

b) Find the Euler parameters associated with that orientation.

c) Develop a method for finding the 3-1-3 angle sequence given only the transformation matrix.

Answers:

$$[R_2]=\begin{bmatrix} C_1C_3-S_1C_2S_3 & S_1C_3+C_1C_2S_3 & S_2S_3 \\ -C_1S_3-S_1C_2C_3 & -S_1S_3+C_1C_2C_3 & S_2C_3 \\ S_1S_2 & -C_1S_2 & C_2 \end{bmatrix}=\begin{bmatrix} 0.728293 & 0.617945 & 0.296198 \\ -0.531121 & 0.235889 & 0.813798 \\ 0.433013 & -0.750000 & 0.500000 \end{bmatrix}$$

$\varepsilon_1=0.498097$ $\varepsilon_2=0.043578$ $\varepsilon_3=0.365998$ $\varepsilon_4=0.784886$

References:

1. T.R. Kane, P.W. Likins, and D.A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983
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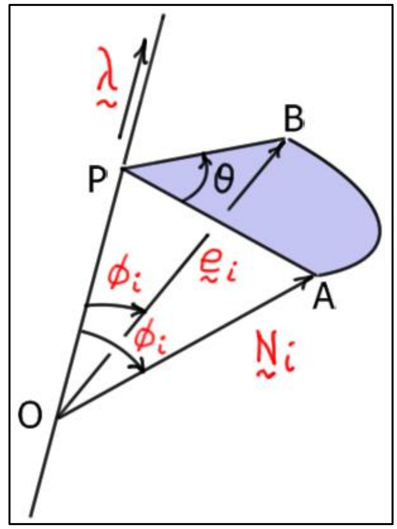
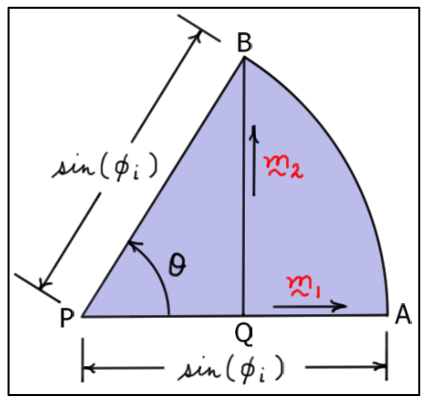
Addendum

Euler's Theorem, Euler Parameters, and Transformation Matrices



To find the elements of the ***transformation matrix*** $[R]$ in terms of the Euler parameters, consider how the body $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ is rotated relative to the base frame $A:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$ using ***Euler's theorem***. Initially the two frames are ***aligned*** so that $\underset{\sim}{e}_i=\underset{\sim}{N}_i \quad (i=1,2,3)$, and then the body is rotated through a single angle θ about an axis (direction) defined by the unit vector $\underset{\sim}{\lambda}$. The diagram shows the rotation of a ***typical unit vector*** in the ***base system*** $\underset{\sim}{N}_i$ into the corresponding unit vector of the ***body-fixed system*** $\underset{\sim}{e}_i$. The angle ϕ_i represents the angle between $\underset{\sim}{\lambda}$ and $\underset{\sim}{N}_i$ which is the same as the angle between $\underset{\sim}{\lambda}$ and $\underset{\sim}{e}_i$.



The elements of any row of $[R]$ can be found by simply expressing the corresponding vector $\underset{\sim}{e}_i$ in terms of the base vector $\underset{\sim}{N}_i$, the angle of rotation θ, and the direction of rotation $\underset{\sim}{\lambda}$. Using the detailed drawing of the circular sector *PAB* to define the unit vectors $\underset{\sim}{m}_1$ and $\underset{\sim}{m}_2$, the unit vector $\underset{\sim}{e}_i$ can be written as follows.

$$\underset{\sim}{e}_i=\underset{\sim}{r}_{P/O}+\underset{\sim}{r}_{Q/P}+\underset{\sim}{r}_{B/Q}=\underbrace{\left[(\underset{\sim}{N}_i\cdot\underset{\sim}{\lambda})\underset{\sim}{\lambda}\right]}_{\underset{\sim}{r}_{P/O}}+\underbrace{\sin(\phi_i)\cos(\theta)\,\underset{\sim}{m}_1}_{\underset{\sim}{r}_{Q/P}}+\underbrace{\sin(\phi_i)\sin(\theta)\,\underset{\sim}{m}_2}_{\underset{\sim}{r}_{B/Q}}$$

$$=\left[(\underset{\sim}{N}_i\cdot\underset{\sim}{\lambda})\underset{\sim}{\lambda}\right]+\cos(\theta)\underbrace{\sin(\phi_i)\,\underset{\sim}{m}_1}_{\underset{\sim}{r}_{A/P}}+\cancel{\sin(\phi_i)}\,\sin(\theta)\underbrace{\left[\frac{\underset{\sim}{\lambda}\times\underset{\sim}{N}_i}{\cancel{\sin(\phi_i)}}\right]}_{\underset{\sim}{m}_2}$$

$$=\left[(\underset{\sim}{N}_i\cdot\underset{\sim}{\lambda})\underset{\sim}{\lambda}\right]+\cos(\theta)\underbrace{\left[\underset{\sim}{N}_i-(\underset{\sim}{N}_i\cdot\underset{\sim}{\lambda})\underset{\sim}{\lambda}\right]}_{\underset{\sim}{r}_{A/P}}+\sin(\theta)(\underset{\sim}{\lambda}\times\underset{\sim}{N}_i)$$

$$\Rightarrow\quad \boxed{\underset{\sim}{e}_i=(1-\cos(\theta))(\underset{\sim}{N}_i\cdot\underset{\sim}{\lambda})\underset{\sim}{\lambda}+\cos(\theta)\,\underset{\sim}{N}_i+\sin(\theta)(\underset{\sim}{\lambda}\times\underset{\sim}{N}_i)}$$

The elements of the first row $(i=1)$ of $[R]$ can now be calculated as follows.

$$
\begin{aligned}
R_{11} &= \underset{\sim}{e}_1 \cdot \underset{\sim}{N}_1 = \left[\left(1-\cos(\theta)\right)\left(\underset{\sim}{N}_1 \cdot \underset{\sim}{\lambda}\right)\underset{\sim}{\lambda} + \cos(\theta)\,\underset{\sim}{N}_1 + \sin(\theta)\left(\underset{\sim}{\lambda}\times\underset{\sim}{N}_1\right)\right]\cdot\underset{\sim}{N}_1 \\
&= \left[\left(1-\cos(\theta)\right)\left(\underset{\sim}{N}_1 \cdot \underset{\sim}{\lambda}\right)\underset{\sim}{\lambda}\cdot\underset{\sim}{N}_1\right] + \left[\cos(\theta)\,\underset{\sim}{N}_1\cdot\underset{\sim}{N}_1\right] + \underbrace{\left[\sin(\theta)\left(\underset{\sim}{\lambda}\times\underset{\sim}{N}_1\right)\cdot\underset{\sim}{N}_1\right]}_{\text{zero}} \\
&= \left[\left(1-\cos(\theta)\right)\lambda_1^2\right] + \left[\cos(\theta)\right] \\
&= \left[2\underbrace{\left(\frac{1-\cos(\theta)}{2}\right)}_{\sin^2(\theta/2)}\frac{\varepsilon_1^2}{\sin^2(\theta/2)}\right] + \left[2\underbrace{\left(\frac{1+\cos(\theta)}{2}\right)}_{\cos^2(\theta/2)}-1\right] \\
&= 2\varepsilon_1^2 + 2\cos^2(\theta/2) - 1 \\
&= 2\varepsilon_1^2 + 2\varepsilon_4^2 - \underbrace{\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2\right)}_{=1}
\end{aligned}
\quad\Rightarrow\quad \boxed{R_{11} = \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2}
$$

$$
\begin{aligned}
R_{12} &= \underset{\sim}{e}_1 \cdot \underset{\sim}{N}_2 = \left[\left(1-\cos(\theta)\right)\left(\underset{\sim}{N}_1 \cdot \underset{\sim}{\lambda}\right)\underset{\sim}{\lambda} + \cos(\theta)\,\underset{\sim}{N}_1 + \sin(\theta)\left(\underset{\sim}{\lambda}\times\underset{\sim}{N}_1\right)\right]\cdot\underset{\sim}{N}_2 \\
&= \left[\left(1-\cos(\theta)\right)\left(\underset{\sim}{N}_1 \cdot \underset{\sim}{\lambda}\right)\underset{\sim}{\lambda}\cdot\underset{\sim}{N}_2\right] + \underbrace{\left[\cos(\theta)\,\underset{\sim}{N}_1\cdot\underset{\sim}{N}_2\right]}_{\text{zero}} + \left[\sin(\theta)\left(\underset{\sim}{\lambda}\times\underset{\sim}{N}_1\right)\cdot\underset{\sim}{N}_2\right] \\
&= \left[\left(1-\cos(\theta)\right)\lambda_1\lambda_2\right] + \left[\sin(\theta)\underbrace{\left(\underset{\sim}{N}_1\times\underset{\sim}{N}_2\right)}_{\underset{\sim}{N}_3}\cdot\underset{\sim}{\lambda}\right] \\
&= \left[2\underbrace{\left(\frac{1-\cos(\theta)}{2}\right)}_{\sin^2(\theta/2)}\frac{\varepsilon_1\varepsilon_2}{\sin^2\left(\frac{\theta}{2}\right)}\right] + \left[\left(2\sin\left(\frac{\theta}{2}\right)\underbrace{\cos\left(\frac{\theta}{2}\right)}_{\varepsilon_4}\right)\left(\frac{\varepsilon_3}{\sin\left(\frac{\theta}{2}\right)}\right)\right]
\end{aligned}
\quad\Rightarrow\quad \boxed{R_{12} = 2\left(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4\right)}
$$

$$
\begin{aligned}
R_{13} &= \underset{\sim}{e}_1 \cdot \underset{\sim}{N}_3 = \left[\left(1-\cos(\theta)\right)\left(\underset{\sim}{N}_1 \cdot \underset{\sim}{\lambda}\right)\underset{\sim}{\lambda} + \cos(\theta)\,\underset{\sim}{N}_1 + \sin(\theta)\left(\underset{\sim}{\lambda}\times\underset{\sim}{N}_1\right)\right]\cdot\underset{\sim}{N}_3 \\
&= \left[\left(1-\cos(\theta)\right)\left(\underset{\sim}{N}_1 \cdot \underset{\sim}{\lambda}\right)\underset{\sim}{\lambda}\cdot\underset{\sim}{N}_3\right] + \underbrace{\left[\cos(\theta)\,\underset{\sim}{N}_1\cdot\underset{\sim}{N}_3\right]}_{\text{zero}} + \left[\sin(\theta)\left(\underset{\sim}{\lambda}\times\underset{\sim}{N}_1\right)\cdot\underset{\sim}{N}_3\right] \\
&= \left[\left(1-\cos(\theta)\right)\lambda_1\lambda_3\right] + \left[\sin(\theta)\underbrace{\left(\underset{\sim}{N}_1\times\underset{\sim}{N}_3\right)}_{-\underset{\sim}{N}_2}\cdot\underset{\sim}{\lambda}\right] \\
&= \left[2\underbrace{\left(\frac{1-\cos(\theta)}{2}\right)}_{\sin^2(\theta/2)}\frac{\varepsilon_1\varepsilon_3}{\sin^2\left(\frac{\theta}{2}\right)}\right] + \left[\left(2\sin\left(\frac{\theta}{2}\right)\underbrace{\cos\left(\frac{\theta}{2}\right)}_{\varepsilon_4}\right)\left(\frac{-\varepsilon_2}{\sin\left(\frac{\theta}{2}\right)}\right)\right]
\end{aligned}
\quad\Rightarrow\quad \boxed{R_{13} = 2\left(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4\right)}
$$

A ***similar process*** can be followed to find the elements of ***rows*** 2 and 3.

Euler's Theorem, Euler Parameters, and Angular Velocity Vectors Components

To find a relationship between the ***Euler parameters***, their ***time derivatives***, and the ***angular velocity components*** of a body, the equation that relates the unit vectors of two different frames can be ***differentiated***. Here, the base-frame is represented by the unit vector set $A:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ and the body-frame is represented by the set $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$. To find the elements of the matrix $[E]$, the ***base-fixed components*** of the angular velocity vector are used.

$$\begin{Bmatrix}\underset{\sim}{e}_1\\\underset{\sim}{e}_2\\\underset{\sim}{e}_3\end{Bmatrix}=[R]\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}\quad\Rightarrow\quad\boxed{\begin{Bmatrix}\dot{\underset{\sim}{e}}_1\\\dot{\underset{\sim}{e}}_2\\\dot{\underset{\sim}{e}}_3\end{Bmatrix}=\left[\dot{R}\right]\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}}$$

where

$$\boxed{\left[\dot{R}\right]=\begin{bmatrix}\dot{R}_{11}&\dot{R}_{12}&\dot{R}_{13}\\\dot{R}_{21}&\dot{R}_{22}&\dot{R}_{23}\\\dot{R}_{31}&\dot{R}_{32}&\dot{R}_{33}\end{bmatrix}}\quad\text{and}\quad\boxed{\begin{aligned}\dot{\underset{\sim}{e}}_i&=\frac{{}^A d}{dt}\left(\underset{\sim}{e}_i\right)={}^A\underset{\sim}{\omega}_B\times\underset{\sim}{e}_i=\begin{vmatrix}\underset{\sim}{N}_1&\underset{\sim}{N}_2&\underset{\sim}{N}_3\\\omega_1&\omega_2&\omega_3\\R_{i1}&R_{i2}&R_{i3}\end{vmatrix}\\&=\left(R_{i3}\omega_2-R_{i2}\omega_3\right)\underset{\sim}{N}_1+\left(R_{i1}\omega_3-R_{i3}\omega_1\right)\underset{\sim}{N}_2+\left(R_{i2}\omega_1-R_{i1}\omega_2\right)\underset{\sim}{N}_3\end{aligned}}$$

Substituting gives

$$\boxed{\begin{bmatrix}\left(R_{13}\omega_2-R_{12}\omega_3\right)&\left(R_{11}\omega_3-R_{13}\omega_1\right)&\left(R_{12}\omega_1-R_{11}\omega_2\right)\\\left(R_{23}\omega_2-R_{22}\omega_3\right)&\left(R_{21}\omega_3-R_{23}\omega_1\right)&\left(R_{22}\omega_1-R_{21}\omega_2\right)\\\left(R_{33}\omega_2-R_{32}\omega_3\right)&\left(R_{31}\omega_3-R_{33}\omega_1\right)&\left(R_{32}\omega_1-R_{31}\omega_2\right)\end{bmatrix}\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}=\begin{bmatrix}\dot{R}_{11}&\dot{R}_{12}&\dot{R}_{13}\\\dot{R}_{21}&\dot{R}_{22}&\dot{R}_{23}\\\dot{R}_{31}&\dot{R}_{32}&\dot{R}_{33}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{N}_1\\\underset{\sim}{N}_2\\\underset{\sim}{N}_3\end{Bmatrix}}$$

It is now convenient to note that the ***coefficient matrix*** on the left side of this result can be written as

$$\boxed{\begin{bmatrix}R_{11}&R_{12}&R_{13}\\R_{21}&R_{22}&R_{23}\\R_{31}&R_{32}&R_{33}\end{bmatrix}\begin{bmatrix}0&\omega_3&-\omega_2\\-\omega_3&0&\omega_1\\\omega_2&-\omega_1&0\end{bmatrix}=\begin{bmatrix}\left(R_{13}\omega_2-R_{12}\omega_3\right)&\left(R_{11}\omega_3-R_{13}\omega_1\right)&\left(R_{12}\omega_1-R_{11}\omega_2\right)\\\left(R_{23}\omega_2-R_{22}\omega_3\right)&\left(R_{21}\omega_3-R_{23}\omega_1\right)&\left(R_{22}\omega_1-R_{21}\omega_2\right)\\\left(R_{33}\omega_2-R_{32}\omega_3\right)&\left(R_{31}\omega_3-R_{33}\omega_1\right)&\left(R_{32}\omega_1-R_{31}\omega_2\right)\end{bmatrix}}$$

Comparing the last two equations, gives

$$\boxed{\begin{bmatrix}R_{11}&R_{12}&R_{13}\\R_{21}&R_{22}&R_{23}\\R_{31}&R_{32}&R_{33}\end{bmatrix}\begin{bmatrix}0&\omega_3&-\omega_2\\-\omega_3&0&\omega_1\\\omega_2&-\omega_1&0\end{bmatrix}=\begin{bmatrix}\dot{R}_{11}&\dot{R}_{12}&\dot{R}_{13}\\\dot{R}_{21}&\dot{R}_{22}&\dot{R}_{23}\\\dot{R}_{31}&\dot{R}_{32}&\dot{R}_{33}\end{bmatrix}}\quad\Rightarrow\quad\boxed{\begin{bmatrix}0&\omega_3&-\omega_2\\-\omega_3&0&\omega_1\\\omega_2&-\omega_1&0\end{bmatrix}=[R]^T\left[\dot{R}\right]}$$

Using the ***double-boxed*** equation, the ***base-fixed components*** of the angular velocity vector can be found in terms of the Euler parameters and their time derivatives. It should be noted that the matrix containing the angular velocity components is a ***skew-symmetric*** matrix.

Calculation of ω_1:

$$\omega_1 = \left[[R]^T [\dot{R}] \right]_{23} = \left[\begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} \dot{R}_{11} & \dot{R}_{12} & \dot{R}_{13} \\ \dot{R}_{21} & \dot{R}_{22} & \dot{R}_{23} \\ \dot{R}_{31} & \dot{R}_{32} & \dot{R}_{33} \end{bmatrix} \right]_{23} = R_{12}\dot{R}_{13} + R_{22}\dot{R}_{23} + R_{32}\dot{R}_{33}$$

$$= \left\{ 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) \cdot \tfrac{d}{dt}\left(2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) \right) \right\} + \left\{ (-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) \cdot \tfrac{d}{dt}\left(2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) \right) \right\}$$
$$+ \left\{ 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \cdot \tfrac{d}{dt}\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \right) \right\}$$

$$= \left\{ 4(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4)(\dot{\varepsilon}_1\varepsilon_3 + \varepsilon_1\dot{\varepsilon}_3 - \dot{\varepsilon}_2\varepsilon_4 - \varepsilon_2\dot{\varepsilon}_4) \right\} + \left\{ 2(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2)(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4) \right\}$$
$$+ \left\{ 4(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)(-\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_2\dot{\varepsilon}_2 + \varepsilon_3\dot{\varepsilon}_3 + \varepsilon_4\dot{\varepsilon}_4) \right\}$$

$$= \left[4\varepsilon_3(\cancel{\varepsilon_1\varepsilon_2} + \varepsilon_3\varepsilon_4) + 2\varepsilon_4(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) - 4\varepsilon_1(\cancel{\varepsilon_2\varepsilon_3} - \varepsilon_1\varepsilon_4) \right]\dot{\varepsilon}_1$$
$$+ \left[-4\varepsilon_4(\cancel{\varepsilon_1\varepsilon_2} + \varepsilon_3\varepsilon_4) + 2\varepsilon_3(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) - 4\varepsilon_2(\varepsilon_2\varepsilon_3 - \cancel{\varepsilon_1\varepsilon_4}) \right]\dot{\varepsilon}_2$$
$$+ \left[4\varepsilon_1(\varepsilon_1\varepsilon_2 + \cancel{\varepsilon_3\varepsilon_4}) + 2\varepsilon_2(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) + 4\varepsilon_3(\varepsilon_2\varepsilon_3 - \cancel{\varepsilon_1\varepsilon_4}) \right]\dot{\varepsilon}_3$$
$$+ \left[-4\varepsilon_2(\varepsilon_1\varepsilon_2 + \cancel{\varepsilon_3\varepsilon_4}) + 2\varepsilon_1(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2) + 4\varepsilon_4(\cancel{\varepsilon_2\varepsilon_3} - \varepsilon_1\varepsilon_4) \right]\dot{\varepsilon}_4$$

$$= \left[2\varepsilon_4 \underbrace{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2)}_{=1} \right]\dot{\varepsilon}_1 + \left[-2\varepsilon_3 \underbrace{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2)}_{=1} \right]\dot{\varepsilon}_2 + \left[2\varepsilon_2 \underbrace{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2)}_{=1} \right]\dot{\varepsilon}_3$$
$$+ \left[-2\varepsilon_1 \underbrace{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2)}_{=1} \right]\dot{\varepsilon}_4$$

$$= 2(\varepsilon_4\dot{\varepsilon}_1 - \varepsilon_3\dot{\varepsilon}_2 + \varepsilon_2\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_4)$$

$$\Rightarrow \boxed{\omega_1 = 2(\varepsilon_4\dot{\varepsilon}_1 - \varepsilon_3\dot{\varepsilon}_2 + \varepsilon_2\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_4) = 2\begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & -\varepsilon_1 \end{bmatrix} \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix}}$$

This result gives the elements of the ***first row*** of the matrix $[E]$. ***Similar*** calculations can be done to find results for the other two components (ω_2, ω_3) and, hence, the ***second*** and ***third*** rows of the matrix. The ***fourth row*** of the matrix is found by simply differentiating the constraint equation $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$ to give

$$\boxed{0 = 2\left(\varepsilon_1\dot{\varepsilon}_1 + \varepsilon_2\dot{\varepsilon}_2 + \varepsilon_3\dot{\varepsilon}_3 + \varepsilon_4\dot{\varepsilon}_4\right)}$$

To find the elements of the matrix $[E']$, the ***body-fixed components*** of the angular velocity vector are used.

$$\begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} \quad \Rightarrow \quad \boxed{\begin{Bmatrix} \dot{\underset{\sim}{e}}_1 \\ \dot{\underset{\sim}{e}}_2 \\ \dot{\underset{\sim}{e}}_3 \end{Bmatrix} = [\dot{R}] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [\dot{R}][R]^T \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}}$$

where

$$\boxed{[\dot{R}] = \begin{bmatrix} \dot{R}_{11} & \dot{R}_{12} & \dot{R}_{13} \\ \dot{R}_{21} & \dot{R}_{22} & \dot{R}_{23} \\ \dot{R}_{31} & \dot{R}_{32} & \dot{R}_{33} \end{bmatrix}}$$

Also,

$$\boxed{\begin{aligned} \dot{\underset{\sim}{e}}_i &= \frac{{}^A d}{dt}\left(\underset{\sim}{e}_i\right) = {}^A\underset{\sim}{\omega}_B \times \underset{\sim}{e}_i = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \omega_1' & \omega_2' & \omega_3' \\ \delta_{i1} & \delta_{i2} & \delta_{i3} \end{vmatrix} \\ &= \left(\delta_{i3}\omega_2' - \delta_{i2}\omega_3'\right)\underset{\sim}{e}_1 + \left(\delta_{i1}\omega_3' - \delta_{i3}\omega_1'\right)\underset{\sim}{e}_2 + \left(\delta_{i2}\omega_1' - \delta_{i1}\omega_2'\right)\underset{\sim}{e}_3 \end{aligned}}$$

Here, the symbol δ_{ij} is equal to ***one*** when $i = j$ and ***zero*** when $i \neq j$. Using the above results gives

$$\boxed{\begin{Bmatrix} \dot{\underset{\sim}{e}}_1 \\ \dot{\underset{\sim}{e}}_2 \\ \dot{\underset{\sim}{e}}_3 \end{Bmatrix} = \begin{bmatrix} 0 & \omega_3' & -\omega_2' \\ -\omega_3' & 0 & \omega_1' \\ \omega_2' & -\omega_1' & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = [\dot{R}][R]^T \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}}$$

Comparing the left and right sides of this result gives

$$\boxed{\boxed{\begin{bmatrix} 0 & \omega_3' & -\omega_2' \\ -\omega_3' & 0 & \omega_1' \\ \omega_2' & -\omega_1' & 0 \end{bmatrix} = [\dot{R}][R]^T}}$$

Using the ***double-boxed*** equation, the ***body-fixed components*** of the ***angular velocity vector*** can be found in terms of the ***Euler parameters*** and ***their time derivatives***. It should be noted that, as before, the matrix containing the angular velocity components is a ***skew-symmetric*** matrix.

To illustrate this process, the calculation of ω_1' is shown below.

$$\omega_1' = \left[\left[\dot{R}\right]\left[R\right]^T\right]_{23} = \left[\begin{bmatrix} \dot{R}_{11} & \dot{R}_{12} & \dot{R}_{13} \\ \dot{R}_{21} & \dot{R}_{22} & \dot{R}_{23} \\ \dot{R}_{31} & \dot{R}_{32} & \dot{R}_{33} \end{bmatrix}\begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix}\right]_{23} = \dot{R}_{21}R_{31} + \dot{R}_{22}R_{32} + \dot{R}_{23}R_{33}$$

$$= \left\{2\left(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4\right)\cdot\tfrac{d}{dt}\left(2\left(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4\right)\right)\right\} + \left\{2\left(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4\right)\cdot\tfrac{d}{dt}\left(-\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 + \varepsilon_4^2\right)\right\}$$
$$+ \left\{\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)\cdot\tfrac{d}{dt}\left(2\left(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4\right)\right)\right\}$$

$$= \left\{4\left(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4\right)\left(\dot{\varepsilon}_1\varepsilon_2 + \varepsilon_1\dot{\varepsilon}_2 - \dot{\varepsilon}_3\varepsilon_4 - \varepsilon_3\dot{\varepsilon}_4\right)\right\} + \left\{4\left(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4\right)\left(-\varepsilon_1\dot{\varepsilon}_1 + \varepsilon_2\dot{\varepsilon}_2 - \varepsilon_3\dot{\varepsilon}_3 + \varepsilon_4\dot{\varepsilon}_4\right)\right\}$$
$$+ \left\{2\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)\left(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4\right)\right\}$$

$$= \left[4\varepsilon_2\left(\cancel{\varepsilon_1\varepsilon_3} + \varepsilon_2\varepsilon_4\right) - 4\varepsilon_1\left(\cancel{\varepsilon_2\varepsilon_3} - \varepsilon_1\varepsilon_4\right) + 2\varepsilon_4\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)\right]\dot{\varepsilon}_1$$
$$+ \left[4\varepsilon_1\left(\varepsilon_1\varepsilon_3 + \cancel{\varepsilon_2\varepsilon_4}\right) + 4\varepsilon_2\left(\varepsilon_2\varepsilon_3 - \cancel{\varepsilon_1\varepsilon_4}\right) + 2\varepsilon_3\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)\right]\dot{\varepsilon}_2$$
$$+ \left[-4\varepsilon_4\left(\cancel{\varepsilon_1\varepsilon_3} + \varepsilon_2\varepsilon_4\right) - 4\varepsilon_3\left(\varepsilon_2\varepsilon_3 - \cancel{\varepsilon_1\varepsilon_4}\right) + 2\varepsilon_2\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)\right]\dot{\varepsilon}_3$$
$$+ \left[-4\varepsilon_3\left(\varepsilon_1\varepsilon_3 + \cancel{\varepsilon_2\varepsilon_4}\right) + 4\varepsilon_4\left(\cancel{\varepsilon_2\varepsilon_3} - \varepsilon_1\varepsilon_4\right) + 2\varepsilon_1\left(-\varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)\right]\dot{\varepsilon}_4$$

$$= \left[2\varepsilon_4\underbrace{\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)}_{=1}\right]\dot{\varepsilon}_1 + \left[2\varepsilon_3\underbrace{\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)}_{=1}\right]\dot{\varepsilon}_2 + \left[-2\varepsilon_2\underbrace{\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)}_{=1}\right]\dot{\varepsilon}_3$$
$$+ \left[-2\varepsilon_1\underbrace{\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right)}_{=1}\right]\dot{\varepsilon}_4$$

$$\Rightarrow \quad \boxed{\omega_1' = 2\left(\varepsilon_4\dot{\varepsilon}_1 + \varepsilon_3\dot{\varepsilon}_2 - \varepsilon_2\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_4\right) = 2\begin{bmatrix}\varepsilon_4 & \varepsilon_3 & -\varepsilon_2 & -\varepsilon_1\end{bmatrix}\begin{Bmatrix}\dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4\end{Bmatrix}}$$

This result gives the elements of the ***first row*** of the matrix $\left[E'\right]$. ***Similar*** calculations can be done to find results for the other two components $\left(\omega_2', \omega_3'\right)$ and, hence, the ***second*** and ***third*** rows of the matrix. As with matrix $\left[E\right]$, the ***fourth row*** of the matrix is found by simply ***differentiating*** the constraint equation $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$.