An Introduction to Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 9

Introduction to Systems with Rolling Constraints

Summary

This unit provides an ***introduction*** to systems having ***rolling constraints***. When the surface of a rigid body is said to be "***rolling***" on another rigid surface (possibly of a second rigid body), it is assumed there is ***no slippage*** between the two surfaces. In this case, even though the two surfaces are ***moving relative to each other***, the ***velocities*** of the ***contact points*** on each of the surfaces must be the ***same***. This unit discusses rolling constraints for two common situations – bodies having a ***single contact point*** and bodies that have a ***line of contact points***. Applications include ball bearings, roller bearings and gearing systems. Two- dimensional systems with rolling constraints are also included to compare and contrast with three-dimensional systems. The coverage is ***not meant to be comprehensive*** but should give the reader a solid introduction to the ***additional complexities*** involved with rolling bodies.

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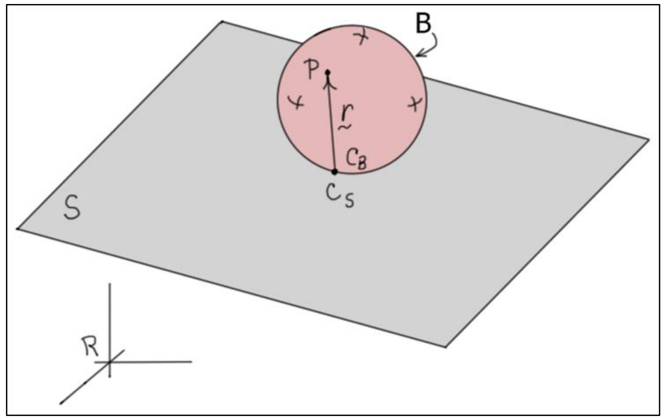
Rolling (without Slipping) in Three Dimensions – Point Contact

When a rigid body rolls (without slipping) on a rigid surface, the surface ***constrains*** its motion. Consider the body B rolling on the rigid surface S as shown. Here,



- S : rigid surface
- B : rigid body
- C_B : contact point on body B
- C_S : contact point on surface S
- R : fixed reference frame

The rigid body B is said to be rolling (without slipping) on surface S if

$$\boxed{{}^{S}\underset{\sim}{v}_{C_B} = \underset{\sim}{0}} \qquad \text{or} \qquad \boxed{{}^{R}\underset{\sim}{v}_{C_B} = {}^{R}\underset{\sim}{v}_{C_S}}$$

The ***velocity*** of other points of B (e.g. P) may be determined by using the formula for relative velocity.

$$\boxed{{}^{R}\underset{\sim}{v}_{P} = {}^{R}\underset{\sim}{v}_{C_B} + {}^{R}\underset{\sim}{v}_{P/C_B} = {}^{R}\underset{\sim}{v}_{C_S} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}_{P/C_B}\right)}$$

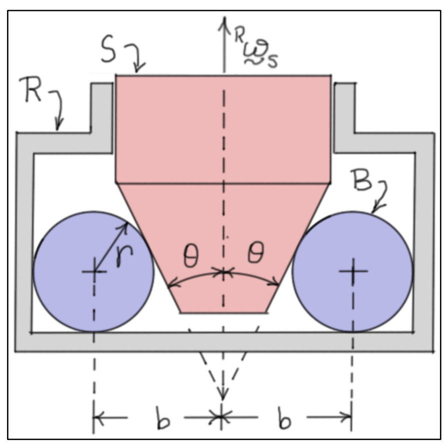
The ***acceleration*** of point P is found by ***direct differentiation***. That is, $\boxed{{}^{R}\underset{\sim}{a}_{P} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P}\right)}$

Notes:

- Even though the ***velocities*** of the contact points C_B and C_S are ***equal*** (i.e. ${}^{R}\underset{\sim}{v}_{C_B} = {}^{R}\underset{\sim}{v}_{C_S}$), the ***accelerations*** of these points are ***not*** (i.e. ${}^{R}\underset{\sim}{a}_{C_B} \neq {}^{R}\underset{\sim}{a}_{C_S}$).
- The ***angular velocity*** ${}^{R}\underset{\sim}{\omega}_{B}$ will, in general, have components that are ***tangent*** and ***normal*** to the contact surface. If ${}^{R}\underset{\sim}{\omega}_{B}$ only has components ***tangent*** to the surface, B is said to have ***pure rolling*** relative to the surface.

Thrust Bearing Example [1]

The diagram shows the geometry of a simple thrust bearing with shaft S, ball bearings B, and race R. The end of the shaft forms a truncated cone as shown. As the shaft is pushed into the ***fixed race*** R, it contacts the surface of the ball bearings B. As S rotates about its axis, it ***rolls*** on the bearings and the bearings ***roll*** on the race. It is assumed that ***no slipping occurs*** at any contact.

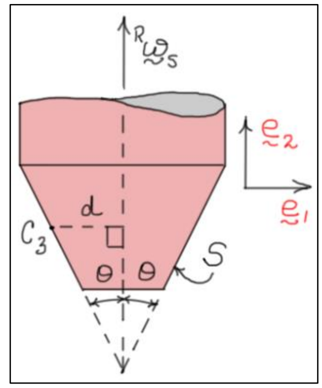


<u>Problem</u>:

Show that to have ***pure rolling*** between shaft S and ball bearing B, it is required to have

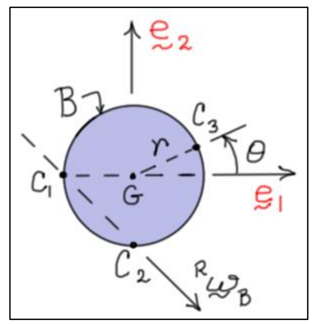
$$\boxed{b = \frac{r\left(1+S_\theta\right)}{C_\theta - S_\theta}}$$

Note that ***pure rolling*** occurs between S and B if ***no slipping*** occurs and if ${}^{B}\underset{\sim}{\omega}_S$ the ***angular velocity*** of S relative to B is ***parallel*** to the ***common tangent plane*** between S and B.



<u>Solution</u>:

1. Consider the separate diagrams below of the shaft and the bearing. Points C_1 and C_2 represent the ***contact points*** between the bearing and the race, and C_3 represents the ***contact point*** between the shaft and the bearing. The ***unit vectors*** $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ represent the ***horizontal*** and ***vertical*** directions in the plane defined by these three points. To complete the unit vector set, a third unit vector is defined as $\underset{\sim}{e}_3 = \underset{\sim}{e}_1 \times \underset{\sim}{e}_2$.



Given this set-up and the no-slip conditions at points C_1, C_2, and C_3, the ***angular velocities*** of the shaft and the bearing may be written as

$$\boxed{{}^{R}\underset{\sim}{\omega}_S = {}^{R}\omega_S\,\underset{\sim}{e}_2} \qquad \boxed{{}^{R}\underset{\sim}{\omega}_B = {}^{R}\omega_B\left(\tfrac{\sqrt{2}}{2}\underset{\sim}{e}_1 - \tfrac{\sqrt{2}}{2}\underset{\sim}{e}_2\right)}$$

2. These two angular velocities can be ***related*** by calculating the ***velocity*** of the contact point C_3. In this process, advantage is taken of the fact that the velocities of the points C_1 and C_2 are ***zero*** due to the no-slip condition with the race. First, take C_3 as a point on the ***bearing*** and write its ***velocity*** as

$${}^{R}\underset{\sim}{v}_{C_3} = \underbrace{{}^{R}\underset{\sim}{v}_{C_1}}_{\text{zero}} + {}^{R}\underset{\sim}{v}_{C_3/C_1} = {}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{C_3/C_1} = {}^{R}\omega_B\left(\tfrac{\sqrt{2}}{2}\underset{\sim}{e}_1 - \tfrac{\sqrt{2}}{2}\underset{\sim}{e}_2\right)\times\left(r\,(1+C_\theta)\underset{\sim}{e}_1 + r\,S_\theta\,\underset{\sim}{e}_2\right)$$

$$\Rightarrow \quad \boxed{{}^{R}\underset{\sim}{v}_{C_3} = \tfrac{\sqrt{2}}{2}\,r\left(S_\theta + (1+C_\theta)\right){}^{R}\omega_B\,\underset{\sim}{e}_3}$$

Then, taking C_3 as a point on the ***shaft***, write its velocity as ${}^{R}\underset{\sim}{v}_{C_3} = d\,{}^{R}\omega_S\,\underset{\sim}{e}_3 \quad\Rightarrow\quad \boxed{{}^{R}\underset{\sim}{v}_{C_3} = \left(b - rC_\theta\right){}^{R}\omega_S\,\underset{\sim}{e}_3}$

Given the ***no-slip condition*** between the bearing and the shaft, these two velocities must be ***equal***. Comparing the two results gives the relationship between the ***angular rates of the shaft and the bearing***. That is,

$$\boxed{{}^{R}\omega_S = \tfrac{\sqrt{2}}{2}\left[\frac{r\left(1+S_\theta+C_\theta\right)}{\left(b - r\,C_\theta\right)}\right]{}^{R}\omega_B}$$

3. The ***angular velocities*** of the shaft and the bearing can also be related using the ***summation rule*** for angular velocities.

$$\boxed{{}^R\underset{\sim}{\omega}_S = {}^R\underset{\sim}{\omega}_B + {}^B\underset{\sim}{\omega}_S}$$

Here, the angular velocity of the bearing is ${}^R\underset{\sim}{\omega}_B = {}^R\omega_B\left(\frac{\sqrt{2}}{2}\underset{\sim}{e}_1 - \frac{\sqrt{2}}{2}\underset{\sim}{e}_2\right)$, the angular velocity of the shaft is ${}^R\underset{\sim}{\omega}_S = \frac{\sqrt{2}}{2}\left[\frac{r(1+S_\theta+C_\theta)}{(b-rC_\theta)}\right]{}^R\omega_B\,\underset{\sim}{e}_2$, and for ***pure rolling*** between S and B, ${}^B\underset{\sim}{\omega}_S = {}^B\omega_S\left(-S_\theta\,\underset{\sim}{e}_1 + C_\theta\,\underset{\sim}{e}_2\right)$. This last equation requires that ${}^B\underset{\sim}{\omega}_S$ to be directed along the ***common tangent line*** of the contacting surfaces. Substituting these results into the ***summation rule*** and separating into ***two scalar equations*** gives

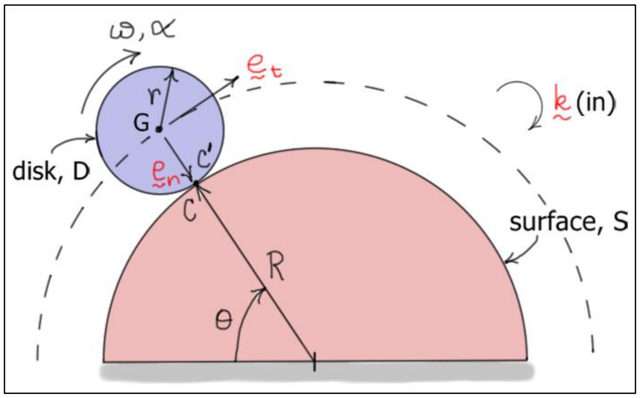
$$\boxed{\begin{aligned} &\tfrac{\sqrt{2}}{2}\,{}^R\omega_B - S_\theta\,{}^B\omega_S = 0 \\ -&\tfrac{\sqrt{2}}{2}\,{}^R\omega_B + C_\theta\,{}^B\omega_S = \tfrac{\sqrt{2}}{2}\left[\frac{r(1+S_\theta+C_\theta)}{(b-rC_\theta)}\right]{}^R\omega_B \end{aligned}}$$

Multiplying the first equation by C_θ and the second by S_θ, ***adding*** the two equations, and ***simplifying*** gives the final result.

$$\boxed{b = \frac{r(1+S_\theta)}{C_\theta - S_\theta}}$$

Rolling (without Slipping) in Two Dimensions – Point Contact

This section discusses an idealized model of a portion of a ***planetary gear system***. The surface S represents the ***sun gear***, and the disk D represents a ***planet gear***. The circular surfaces of S and D represent the ***pitch circles*** of these gears.



Rolling on a Fixed Surface

If a rigid body ***rolls*** (without slipping) on a ***fixed surface***, the point that is in contact with the surface has ***zero velocity***. For example, consider a ***circular*** disk D that rolls on the fixed ***circular*** surface S as shown. Because C' is in contact with the point C on the fixed surface, its ***velocity*** is ***zero***.

Using this result, the ***velocity*** of G the center of the disk can be calculated using the relative velocity equation.

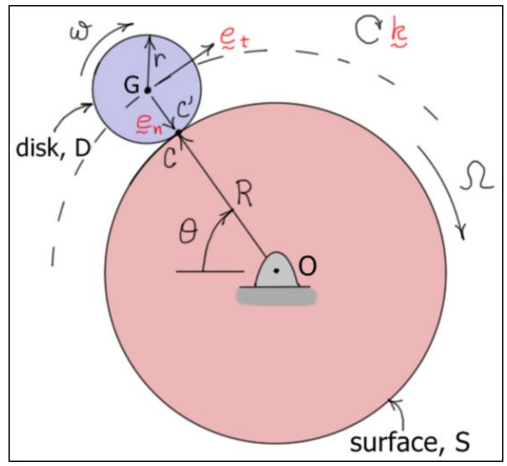
$${}^R\underset{\sim}{v}_G = \underbrace{{}^R\underset{\sim}{v}_{C'}}_{\text{zero}} + {}^R\underset{\sim}{v}_{G/C'} = {}^R\underset{\sim}{v}_{G/C'} = {}^R\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{G/C'} = \omega\,\underset{\sim}{k}\times\left(-r\,\underset{\sim}{e}_n\right) \quad\Rightarrow\quad \boxed{{}^R\underset{\sim}{v}_G = v\,\underset{\sim}{e}_t = r\,\omega\,\underset{\sim}{e}_t}$$

The acceleration of G is calculated by ***differentiating*** the expression for ${}^R\underset{\sim}{v}_G$.

$$
{}^{R}\underset{\sim}{a}_G = \frac{{}^{R}d}{dt}\left(r\,\omega\,\underset{\sim}{e}_t\right) = r\,\dot{\omega}\,\underset{\sim}{e}_t + r\,\omega\,\dot{\underset{\sim}{e}}_t = r\,\alpha\,\underset{\sim}{e}_t + r\,\omega\left(\dot{\theta}\underset{\sim}{k}\times\underset{\sim}{e}_t\right) = r\,\alpha\,\underset{\sim}{e}_t + r\,\omega\left(\frac{v}{R+r}\,\underset{\sim}{e}_n\right)
$$

$$
= r\,\alpha\,\underset{\sim}{e}_t + r\,\omega\left(\frac{r\,\omega}{R+r}\,\underset{\sim}{e}_n\right) \quad\Rightarrow\quad \boxed{{}^{R}\underset{\sim}{a}_G = r\,\alpha\,\underset{\sim}{e}_t + \left(\frac{r^2\omega^2}{R+r}\right)\underset{\sim}{e}_n}
$$

This last result is ***expected*** because the ***tangential acceleration*** of G is the rate of change of its ***speed*** and the ***normal acceleration*** is its ***velocity squared*** divided by the ***radius of curvature*** $\left(v^2/\rho = v^2/(R+r)\right)$.



Rolling on a Moving Surface

If a rigid body ***rolls*** (without slipping) on a ***moving surface***, then the velocities of the two contact points C and C' must be the same (i.e. $\underset{\sim}{v}_{C'} = \underset{\sim}{v}_C$). For example, consider a ***circular*** disk D that rolls on the rotating ***circular*** surface S as shown in the figure. As before, the ***velocity*** of the mass center G can be calculated using the relative velocity equation.

$$
{}^{R}\underset{\sim}{v}_G = {}^{R}\underset{\sim}{v}_{C'} + {}^{R}\underset{\sim}{v}_{G/C'} = {}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{r}_{C/O} + {}^{R}\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{G/C'} = \Omega\underset{\sim}{k}\times\left(-R\underset{\sim}{e}_n\right) + \omega\underset{\sim}{k}\times\left(-r\underset{\sim}{e}_n\right)
$$

$$
\Rightarrow\quad \boxed{{}^{R}\underset{\sim}{v}_G = v\,\underset{\sim}{e}_t = \left(R\,\Omega + r\,\omega\right)\underset{\sim}{e}_t}
$$

Again, the ***acceleration*** of G is found by ***differentiating*** the velocity vector.

$$
{}^{R}\underset{\sim}{a}_G = \frac{{}^{R}d}{dt}\left(R\,\Omega + r\,\omega\right)\underset{\sim}{e}_t = \left(R\,\dot{\Omega} + r\,\dot{\omega}\right)\underset{\sim}{e}_t + \left(R\,\Omega + r\,\omega\right)\dot{\underset{\sim}{e}}_t
$$

$$
= \left(R\,\dot{\Omega} + r\,\dot{\omega}\right)\underset{\sim}{e}_t + \left(R\,\Omega + r\,\omega\right)\left(\dot{\theta}\underset{\sim}{k}\times\underset{\sim}{e}_t\right)
$$

$$
= \left(R\dot{\Omega} + r\dot{\omega}\right)\underset{\sim}{e}_t + \left(R\Omega + r\omega\right)\left(\frac{v}{R+r}\,\underset{\sim}{e}_n\right) \quad\Rightarrow\quad \boxed{{}^{R}\underset{\sim}{a}_G = \left(R\,\dot{\Omega} + r\,\dot{\omega}\right)\underset{\sim}{e}_t + \left(\frac{\left(R\Omega + r\omega\right)^2}{R+r}\right)\underset{\sim}{e}_n}
$$

Again, this last result is ***expected*** because the ***tangential acceleration*** of G is the rate of change of its ***speed*** and the ***normal acceleration*** is its ***velocity squared*** divided by the ***radius of curvature*** $\left(v^2/\rho = v^2/(R+r)\right)$.

Notes:

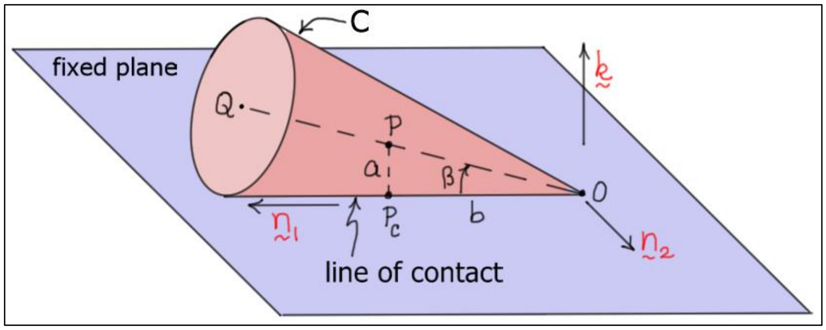
- Even though the ***velocities*** of the contact points are equal (i.e. ${}^{R}\underset{\sim}{v}_{C_B} = {}^{R}\underset{\sim}{v}_{C_S}$ and ${}^{R}\underset{\sim}{v}_{C'} = {}^{R}\underset{\sim}{v}_C$), the ***accelerations*** of these points are ***not*** (i.e. ${}^{R}\underset{\sim}{a}_{C_B} \neq {}^{R}\underset{\sim}{a}_{C_S}$ and ${}^{R}\underset{\sim}{a}_{C'} \neq {}^{R}\underset{\sim}{a}_C$). However, for ***two dimensional systems***, it can be shown that the ***components*** of these ***accelerations*** along the ***tangential direction*** $\underset{\sim}{e}_t$ are ***equal*** (i.e. ${}^{R}\underset{\sim}{a}_{C'}\cdot\underset{\sim}{e}_t = {}^{R}\underset{\sim}{a}_C\cdot\underset{\sim}{e}_t$).

- In practice, planetary gear systems have a ***separate component*** that keeps the ***centers*** of its planetary gears at a ***fixed distance*** from the center of the sun gear so the planetary gears will ***remain in contact*** with the sun gear at all times.
- Planetary gear systems may also have a ring gear that circumscribes the planetary gears. In this case, the sun and planetary gears are external gears (teeth on the outer surface of the gear), the ring gear is an internal gear (teeth on the inner surface of the gear). The ring gear may be stationary or it may also rotate.

Rolling (without slipping) – Line Contact

Cone Rolling on a Flat Plane

Consider a ***right circular cone*** *C* rolling without slipping on a ***flat plane***. If the plane is ***fixed***, then all the points on the line of contact of the cone must have ***zero velocity***, and the cone will roll in a ***circular path*** with the point *O* remaining fixed.



To analyze the kinematics of the cone as it rolls, first define the reference frame $S:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{k})$. The unit vectors of *S* are directed as follows: $\underset{\sim}{n}_1$ is directed ***along the line of contact***, $\underset{\sim}{k}$ is ***normal*** to the plane, and $\underset{\sim}{n}_2=\underset{\sim}{k}\times\underset{\sim}{n}_1$. Because all the points on the contact line have zero velocity, the ***angular velocity*** of the ***cone*** must be along the $\underset{\sim}{n}_1$ direction, that is, $\boxed{{}^R\underset{\sim}{\omega}_C=\omega\underset{\sim}{n}_1}$.

The ***velocity*** of any point of the cone may be calculated using the ***relative velocity*** equation. For example, suppose P_C is a point on the contact line, and *P* is a point along the centerline of the cone, a distance *a* directly above P_C (distance *a* is measured ***normal*** to the plane). Then, the velocity of *P* may be calculated as

$$
{}^R\underset{\sim}{v}_P={}^R\underset{\sim}{v}_{P_C}+{}^R\underset{\sim}{v}_{P/P_C}={}^R\underset{\sim}{v}_{P_C}+\left({}^R\underset{\sim}{\omega}_C\times{}^R\underset{\sim}{r}_{P/P_C}\right)=\omega\underset{\sim}{n}_1\times a\underset{\sim}{k}\quad\Rightarrow\quad\boxed{{}^R\underset{\sim}{v}_P=-a\,\omega\,\underset{\sim}{n}_2}
$$

Now, consider a set of points *P* from the vertex at *O* to the center of the cone's base at *Q*. As point *P* is moved from *O* to *Q*, the length *a* (and, hence, the velocity of the point *P*) ***increases linearly***. Consequently, the cone rolls in a ***circular path*** with the point *O* remaining ***fixed***.

The ***angular acceleration*** of the cone is found by ***differentiating*** ${}^R\underset{\sim}{\omega}_C$.

$$
{}^R\underset{\sim}{\alpha}_C=\frac{{}^Rd}{dt}\left(\omega\underset{\sim}{n}_1\right)=\dot{\omega}\underset{\sim}{n}_1+\omega\dot{\underset{\sim}{n}}_1=\dot{\omega}\underset{\sim}{n}_1+\omega\left({}^R\underset{\sim}{\omega}_S\times\underset{\sim}{n}_1\right)=\dot{\omega}\underset{\sim}{n}_1+\omega\left(\Omega\underset{\sim}{k}\times\underset{\sim}{n}_1\right)\quad\Rightarrow\quad\boxed{{}^R\underset{\sim}{\alpha}_C=\dot{\omega}\underset{\sim}{n}_1+\omega\Omega\underset{\sim}{n}_2}
$$

Here, ${}^R\underset{\sim}{\omega}_S=\Omega\underset{\sim}{k}$ represents the angular velocity of the frame *S* relative to the fixed plane. Hence, it describes the ***rotation rate of the line of contact*** within the plane.

Because the cone is rolling (without slipping), the angular rates ω and Ω are ***not independent***. To find a ***relationship between the two rates***, consider again a point *P* on the centerline of the cone. As noted above, the velocity of *P* may be written as ${}^{R}\underset{\sim}{v}_{P} = -a\,\omega\,\underset{\sim}{n}_2$. However, it can also be written as ${}^{R}\underset{\sim}{v}_{P} = b\,\Omega\,\underset{\sim}{n}_2$ where b represents the distance from P_C to *O*. Comparing these two expressions yields the result, $\boxed{\Omega = -\left(a/b\right)\omega = -\omega\,\tan(\beta)}$, where β represents the ***half angle*** of the cone. Using this result, ${}^{R}\underset{\sim}{\alpha}_{C}$ the ***angular acceleration*** of the cone may be written as $\boxed{{}^{R}\underset{\sim}{\alpha}_{C} = \dot{\omega}\,\underset{\sim}{n}_1 - \omega^2\tan(\beta)\,\underset{\sim}{n}_2}$. The ***acceleration*** of *P* is found by ***differentiating*** the velocity ${}^{R}\underset{\sim}{v}_{P}$.

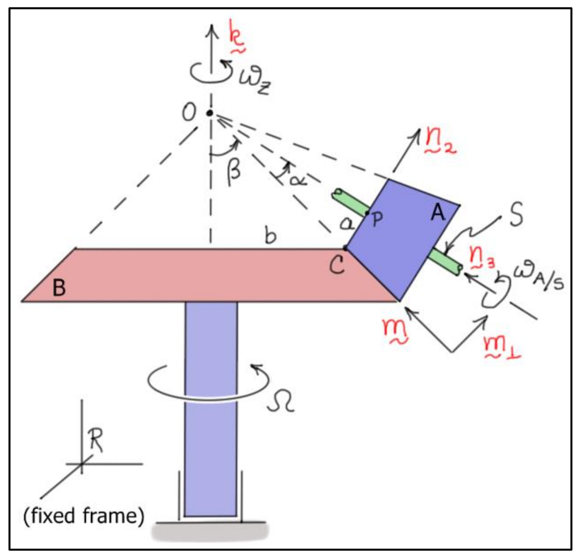
$$ {}^{R}\underset{\sim}{a}_{P} = \frac{{}^{R}d}{dt}\left(-a\,\omega\,\underset{\sim}{n}_2\right) = -a\left(\dot{\omega}\,\underset{\sim}{n}_2 + \omega\,\dot{\underset{\sim}{n}}_2\right) = -a\left(\dot{\omega}\,\underset{\sim}{n}_2 + \omega\left(\Omega\,\underset{\sim}{k}\times\underset{\sim}{n}_2\right)\right) = -a\left(\dot{\omega}\,\underset{\sim}{n}_2 - \omega\,\Omega\,\underset{\sim}{n}_1\right) $$

or

$$ \boxed{{}^{R}\underset{\sim}{a}_{P} = -a\,\dot{\omega}\,\underset{\sim}{n}_2 - a\,\omega^2\tan(\beta)\,\underset{\sim}{n}_1} $$

<u>Beveled Gears</u>

Beveled gears are a practical example of bodies that roll with a line of contact. Each gear may be thought of as a truncated ***cone***, so two ***contacting gears*** may be thought of as two cones ***rolling on each other***. Consider, for example, the system of two beveled gears shown. Gear *B* is ***keyed*** to a vertical shaft that rotates with speed Ω about the $\underset{\sim}{k}$ direction. Gear *A* rolls on gear *B* and rotates ***freely*** on the axle *S*. The axle rotates at a rate ω_z about the $\underset{\sim}{k}$ direction (pivoting about point *O*), and gear *A* rotates ***relative*** to the axle at a rate $\omega_{A/S}$.



To analyze the kinematics of this system, first define a unit vector set fixed in the axle as $S:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3)$. The direction $\underset{\sim}{n}_3$ is pointed along the axle towards *O*, the direction $\underset{\sim}{n}_2$ is perpendicular to the axle as shown in the diagram, and the direction $\underset{\sim}{n}_1 = \underset{\sim}{n}_2\times\underset{\sim}{n}_3$. The angles α and β are the half angles of the two cones (as shown), and the sum of these angles is defined here as angle $\gamma = \alpha + \beta$. Point *C* is on the contact line between the two gears, and point *P* is a point on the centerline of axle *S*.

The concept of relative velocity can be used to find a relationship between the angular rates Ω, ω_z, and $\omega_{A/S}$. If point *C* is the contact point on gear *A*, the velocity ${}^{R}\underset{\sim}{v}_{C}$ may be calculated as follows

$$ {}^R\underset{\sim}{v}_C = {}^R\underset{\sim}{v}_P + {}^R\underset{\sim}{v}_{C/P} $$

where

$$ {}^R\underset{\sim}{v}_P = -\left(b + a\,C_\gamma\right)\omega_z\,\underset{\sim}{n}_1 \quad \ldots \text{ (P moves in a circle around the vertical axis with radius } r = b + a\,C_\gamma) $$

$$ {}^R\underset{\sim}{v}_{C/P} = {}^R\underset{\sim}{\omega}_A \times \underset{\sim}{r}_{C/P} = \left({}^R\underset{\sim}{\omega}_S + {}^S\underset{\sim}{\omega}_A\right)\times\left(-a\,\underset{\sim}{n}_2\right) = (\omega_z\,\underset{\sim}{k} + \omega_{A/S}\,\underset{\sim}{n}_3)\times\left(-a\,\underset{\sim}{n}_2\right) = a\left(\omega_{A/S} + \omega_z\,C_\gamma\right)\underset{\sim}{n}_1 $$

Considering C to be the contact point on gear B, the velocity of C may also be written as ${}^R\underset{\sim}{v}_C = -b\,\Omega\,\underset{\sim}{n}_1$.

Equating the velocities of the two contact points and simplifying gives

$$ \boxed{\omega_{A/S} = (b/a)\left(\omega_z - \Omega\right)} $$

<u>Check on the Relative Angular Velocities</u>: ${}^B\underset{\sim}{\omega}_A$ and ${}^A\underset{\sim}{\omega}_B$

As noted above, when a cone rolls on a fixed plane, the angular velocity of the cone relative to the plane is directed along the line of contact. Similarly, when ***two beveled gears roll on each other***, the ***relative angular velocities*** ${}^B\underset{\sim}{\omega}_A$ and ${}^A\underset{\sim}{\omega}_B$ must be ***directed along the line of contact***. In the diagram, the unit vector $\underset{\sim}{m}$ is directed along the line of contact, and the unit vector $\underset{\sim}{m}_\perp = \underset{\sim}{n}_1 \times \underset{\sim}{m}$ is directed perpendicular to the line of contact.

$$ \boxed{{}^B\underset{\sim}{\omega}_A = \omega_{A/B}\,\underset{\sim}{m} = -{}^A\underset{\sim}{\omega}_B} $$

This result can be ***verified*** by computing the angular velocities and using the results found above. For example, using the ***angular velocity summation rule***, ${}^B\underset{\sim}{\omega}_A$ may be calculated as follows

$$ \begin{aligned} {}^B\underset{\sim}{\omega}_A &= {}^R\underset{\sim}{\omega}_A - {}^R\underset{\sim}{\omega}_B = (\omega_z\,\underset{\sim}{k} + \omega_{A/S}\,\underset{\sim}{n}_3) - \Omega\,\underset{\sim}{k} = (\omega_z - \Omega)\,\underset{\sim}{k} + \omega_{A/S}\,\underset{\sim}{n}_3 \\ &= (\omega_z - \Omega)\left(C_\beta\,\underset{\sim}{m} + S_\beta\,\underset{\sim}{m}_\perp\right) + \tfrac{b}{a}(\omega_z - \Omega)\left(C_\alpha\,\underset{\sim}{m} - S_\alpha\,\underset{\sim}{m}_\perp\right) \\ &= (\omega_z - \Omega)\left((C_\beta + \tfrac{b}{a}C_\alpha)\,\underset{\sim}{m} + (S_\beta - \tfrac{b}{a}S_\alpha)\,\underset{\sim}{m}_\perp\right) \end{aligned} $$

where

$$ \boxed{\frac{b}{a} = \frac{b}{\left|\overline{OC}\right|}\cdot\frac{\left|\overline{OC}\right|}{a} = \frac{S_\beta}{S_\alpha}} \qquad \boxed{C_\beta + \tfrac{b}{a}C_\alpha = C_\beta + \frac{S_\beta C_\alpha}{S_\alpha} = \frac{C_\beta S_\alpha + S_\beta C_\alpha}{S_\alpha} = \frac{S_{\alpha+\beta}}{S_\alpha}} $$

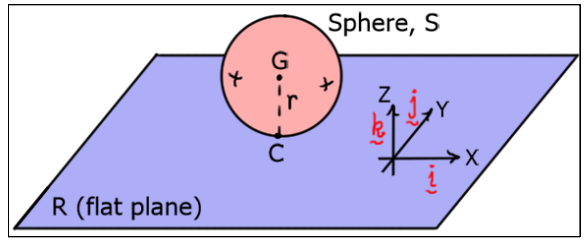
$$ \boxed{S_\beta - \tfrac{b}{a}S_\alpha = S_\beta - \frac{S_\beta S_\alpha}{S_\alpha} = \frac{S_\beta S_\alpha - S_\beta S_\alpha}{S_\alpha} = 0} $$

Substituting these results into the above equation gives the final result showing that the relative angular velocities of the two gears are along the line of contact between them.

$$ \boxed{{}^B\underset{\sim}{\omega}_A = (\omega_z - \Omega)\left(\frac{S_{\alpha+\beta}}{S_\alpha}\right)\underset{\sim}{m}} $$

Exercises

9.1 The sphere ***rolls without slipping*** on the flat plane. The Z-axis is ***normal*** to the plane, C is the point on the sphere in ***contact*** with the plane, and G is the ***center*** of the sphere. The angular velocity and angular acceleration of the sphere are given as



$$ {}^{R}\underset{\sim}{\omega}_{S} = \omega_x \underset{\sim}{i} + \omega_y \underset{\sim}{j} + \omega_z \underset{\sim}{k} \quad \text{and} \quad {}^{R}\underset{\sim}{\alpha}_{S} = \dot{\omega}_x \underset{\sim}{i} + \dot{\omega}_y \underset{\sim}{j} + \dot{\omega}_z \underset{\sim}{k} $$

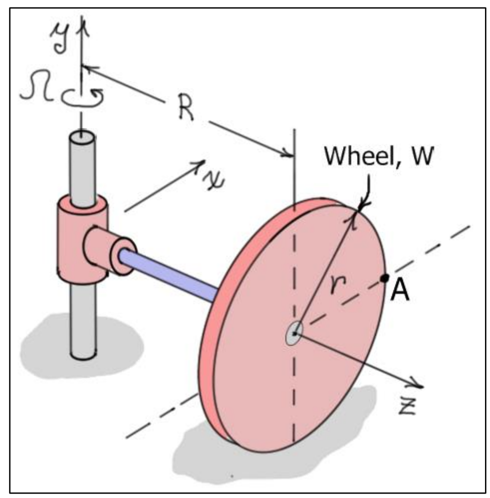
Find ${}^{R}\underset{\sim}{v}_{G}$ the velocity of G, ${}^{R}\underset{\sim}{a}_{G}$ the acceleration of G, and ${}^{R}\underset{\sim}{a}_{C}$ the acceleration of C.

Answers:

$$ {}^{R}\underset{\sim}{v}_{G} = r\left(\omega_y \underset{\sim}{i} - \omega_x \underset{\sim}{j}\right) \qquad {}^{R}\underset{\sim}{a}_{G} = r\left(\dot{\omega}_y \underset{\sim}{i} - \dot{\omega}_x \underset{\sim}{j}\right) $$

$$ {}^{R}\underset{\sim}{a}_{C} = \left(-r\,\omega_x \omega_z\right)\underset{\sim}{i} + \left(-r\,\omega_y \omega_z\right)\underset{\sim}{j} + r\left(\omega_x^2 + \omega_y^2\right)\underset{\sim}{k} $$

9.2 The system shown consists of a vertical column, a horizontal axle, and a wheel of radius r. The horizontal arm rotates at a ***constant*** rate Ω, and the wheel (W) rolls without slipping in a circular arc. Find $\underset{\sim}{\omega}_W$ and $\underset{\sim}{\alpha}_W$ the angular velocity and angular acceleration of the wheel relative to a fixed frame, and find $\underset{\sim}{v}_A$ and $\underset{\sim}{a}_A$ the velocity and acceleration of point A.

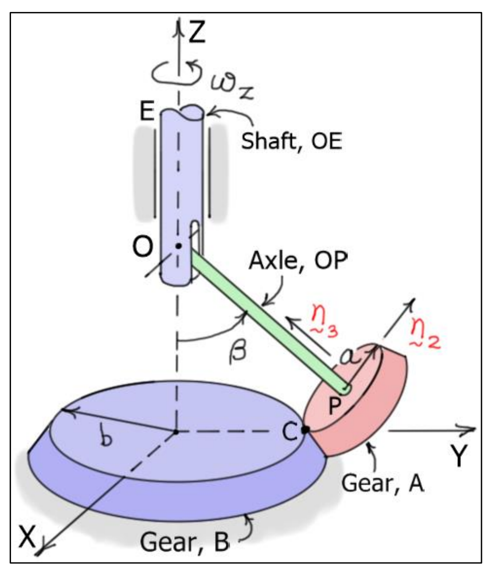


Answers:

$$ {}^{R}\underset{\sim}{\omega}_{W} = \Omega\,\underset{\sim}{j} - (R/r)\Omega\,\underset{\sim}{k} \qquad {}^{R}\underset{\sim}{\alpha}_{W} = -(R/r)\Omega^2\,\underset{\sim}{i} $$

$$ {}^{R}\underset{\sim}{v}_{A} = R\Omega\underset{\sim}{i} - R\Omega\underset{\sim}{j} - r\Omega\underset{\sim}{k} \qquad {}^{R}\underset{\sim}{a}_{A} = -\left(\frac{R^2 + r^2}{r}\right)\Omega^2\underset{\sim}{i} - R\Omega^2\underset{\sim}{k} $$

9.3 In the system shown, ***beveled gear*** A rolls on ***beveled gear*** B. As A rolls on B it ***spins*** about the ***axle*** OP which is pinned to the vertical shaft OE at O. If OE rotates at a ***constant*** angular velocity ω_z (rad/sec), find ${}^{OP}\underset{\sim}{\omega}_{A}$ the angular velocity of gear A relative to the axle OP, ${}^{R}\underset{\sim}{\omega}_{A}$ the ***angular velocity*** of gear A, ${}^{R}\underset{\sim}{\alpha}_{A}$ the ***angular acceleration*** of gear A, and $\underset{\sim}{a}_C$ the ***acceleration*** of the ***contact point*** C of gear A.



Answers:

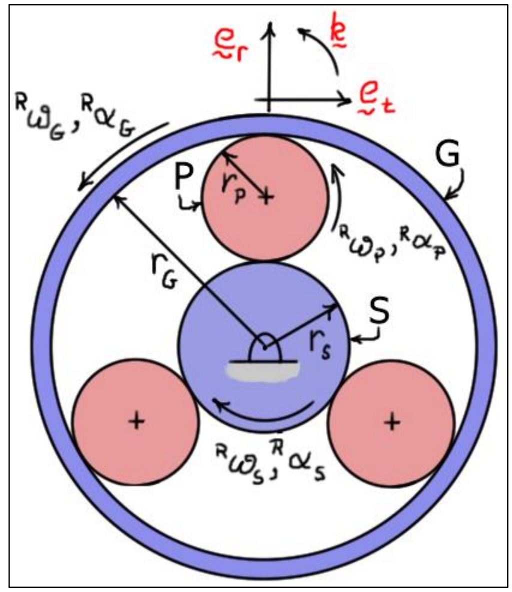
$$ {}^{OP}\underset{\sim}{\omega}_{A} = \left(\tfrac{b}{a}\right)\omega_z\,\underset{\sim}{n}_3 $$

$$ {}^{R}\underset{\sim}{\omega}_{A} = \omega_z\,\underset{\sim}{k} + (\tfrac{b}{a})\,\omega_z\,\underset{\sim}{n}_3 = \omega_z\,S_\beta\,\underset{\sim}{n}_2 + \omega_z\,(\tfrac{b}{a} + C_\beta)\,\underset{\sim}{n}_3 $$

$$ {}^{R}\underset{\sim}{\alpha}_{A} = \left(\tfrac{b}{a}\right)\omega_z^2\,S_\beta\,\underset{\sim}{n}_1 \qquad {}^{R}\underset{\sim}{a}_{C} = \omega_z^2\left[\left(\tfrac{b}{a}\right)(b + aC_\beta)\,\underset{\sim}{n}_2 - b\,S_\beta\,\underset{\sim}{n}_3\right] $$

9.4 The figure depicts a ***planetary gear system*** moving in a fixed frame R. The ***ring gear*** G has radius r_G, angular velocity ${}^R\omega_G$, and angular acceleration ${}^R\alpha_G$. The ***sun gear*** S has radius r_S, angular velocity ${}^R\omega_S$, and angular acceleration ${}^R\alpha_S$. ***Show*** that ${}^R\omega_P$ the angular velocity of the ***planet gear*** and ${}^R\alpha_P$ the angular acceleration of the ***planet gear*** may be written as

$$ {}^R\omega_P = \left(r_G\,{}^R\omega_G + r_S\,{}^R\omega_S \right) \Big/ 2r_P $$
$$ {}^R\alpha_P = \left(r_G\,{}^R\alpha_G + r_S\,{}^R\alpha_S \right) \Big/ 2r_P $$



References:

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