Introduction to Systems with Rolling Constraints

Summary

This unit provides an introduction to systems having rolling constraints. When the surface of a rigid body is said to be “rolling” on another rigid surface (possibly of a second rigid body), it is assumed there is no slippage between the two surfaces. In this case, even though the two surfaces are moving relative to each other, the velocities of the contact points on each of the surfaces must be the same. This unit discusses rolling constraints for two common situations – bodies having a single contact point and bodies that have a line of contact points. Applications include ball bearings, roller bearings and gearing systems. Two-dimensional systems with rolling constraints are also included to compare and contrast with three-dimensional systems. The coverage is not meant to be comprehensive but should give the reader a solid introduction to the additional complexities involved with rolling bodies.

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Rolling (without Slipping) in Three Dimensions – Point Contact

When a rigid body rolls (without slipping) on a rigid surface, the surface constrains its motion. Consider the body $B$ rolling on the rigid surface $S$ as shown. Here,

- $S$: rigid surface
- $B$: rigid body
- $C_B$: contact point on body $B$
- $C_S$: contact point on surface $S$
- $R$: fixed reference frame

The rigid body $B$ is said to be rolling (without slipping) on surface $S$ if

$$ S \dot{y}_{C_B} = 0 \quad \text{or} \quad R \dot{y}_{C_B} \cdot R \dot{y}_{C_B} = R \dot{y}_{C_S} \cdot R \dot{y}_{C_S} $$

The velocity of other points of $B$ (e.g. $P$) may be determined by using the formula for relative velocity.

$$ R \dot{y}_P = R \dot{y}_{C_B} + R \dot{y}_{P/C_B} = R \dot{y}_{C_S} + \left( R \omega_B \times R \dot{y}_{P/C_S} \right) $$

The acceleration of point $P$ is found by direct differentiation. That is,

$$ R \ddot{a}_P = \frac{d}{dt} \left( R \dot{y}_P \right) $$

Notes:

- Even though the velocities of the contact points $C_B$ and $C_S$ are equal (i.e. $R \dot{y}_{C_B} = R \dot{y}_{C_S}$), the accelerations of these points are not (i.e. $R \ddot{a}_{C_B} \neq R \ddot{a}_{C_S}$).
- The angular velocity $R \omega_B$ will, in general, have components that are tangent and normal to the contact surface. If $R \omega_B$ only has components tangent to the surface, $B$ is said to have pure rolling relative to the surface.

Thrust Bearing Example [1]

The diagram shows the geometry of a simple thrust bearing with shaft $S$, ball bearings $B$, and race $R$. The end of the shaft forms a truncated cone as shown. As the shaft is pushed into the fixed race $R$, it contacts the surface of the ball bearings $B$. As $S$ rotates about its axis, it rolls on the bearings and the bearings roll on the race. It is assumed that no slipping occurs at any contact.

Problem:

Show that to have pure rolling between shaft $S$ and ball bearing $B$, it is required to have
\[ b = \frac{r(1 + S_\theta)}{C_\theta - S_\theta} \]

Note that pure rolling occurs between \( S \) and \( B \) if no slipping occurs and if \( \dot{R}\omega_S \) the angular velocity of \( S \) relative to \( B \) is parallel to the common tangent plane between \( S \) and \( B \).

**Solution:**

1. Consider the separate diagrams below of the shaft and the bearing. Points \( C_1 \) and \( C_2 \) represent the contact points between the bearing and the race, and \( C_3 \) represents the contact point between the shaft and the bearing.

   The unit vectors \( \varepsilon_1 \) and \( \varepsilon_2 \) represent the horizontal and vertical directions in the plane defined by these three points. To complete the unit vector set, a third unit vector is defined as \( \varepsilon_3 = \varepsilon_1 \times \varepsilon_2 \).

   Given this set-up and the no-slip conditions at points \( C_1 \), \( C_2 \), and \( C_3 \), the angular velocities of the shaft and the bearing may be written as

   \[ \dot{R}\omega_S = \dot{R}\omega_S \varepsilon_2 \]
   \[ \dot{R}\omega_B = \dot{R}\omega_B \left( \frac{\sqrt{2}}{2} \varepsilon_1 - \frac{\sqrt{2}}{2} \varepsilon_2 \right) \]

2. These two angular velocities can be related by calculating the velocity of the contact point \( C_3 \). In this process, advantage is taken of the fact that the velocities of the points \( C_1 \) and \( C_2 \) are zero due to the no-slip condition with the race. First, take \( C_3 \) as a point on the bearing and write its velocity as

   \[ \dot{R}_{C_3} = \dot{R}_{C_3} + \dot{R}_{C_1/C_2} = \dot{R}_{C_1/C_2} = \dot{R}_{C_1/C_2} \left( \frac{\sqrt{2}}{2} \varepsilon_1 - \frac{\sqrt{2}}{2} \varepsilon_2 \right) \times \left( r (1 + C_\theta) \varepsilon_1 + r S_\theta \varepsilon_2 \right) \]

   \[ \Rightarrow \dot{R}_{C_3} = \frac{\sqrt{2}}{2} r \left( S_\theta + (1 + C_\theta) \right) \dot{R}_{C_3} \varepsilon_3 \]

   Then, taking \( C_3 \) as a point on the shaft, write its velocity as \( \dot{R}_{C_3} = d \dot{R}\omega_S \varepsilon_3 \Rightarrow \dot{R}_{C_3} = (b - r C_\theta) \dot{R}\omega_S \varepsilon_3 \)

   Given the no-slip condition between the bearing and the shaft, these two velocities must be equal. Comparing the two results gives the relationship between the angular rates of the shaft and the bearing. That is,

   \[ \dot{R}\omega_S = \frac{\sqrt{2}}{2} \left[ \frac{r (1 + S_\theta + C_\theta)}{(b - r C_\theta)} \right] \dot{R}\omega_B \]

3. The angular velocities of the shaft and the bearing can also be related using the summation rule for angular velocities.
Here, the angular velocity of the bearing is $\omega_B = \omega_B \left( \frac{\sqrt{e_1}}{2} - \frac{\sqrt{e_2}}{2} \right)$, the angular velocity of the shaft is 

$$\omega_S = \frac{\sqrt{e_2}}{2} \left[ \frac{r(1+S_\theta+C_\theta)}{(b-rC_\theta)} \right] \omega_B \varepsilon_2,$$

and for pure rolling between $S$ and $B$, $\omega_S = \omega_S \left( -S_\theta \varepsilon_1 + C_\theta \varepsilon_2 \right)$. This last equation requires that $\omega_S$ to be directed along the common tangent line of the contacting surfaces.

Substituting these results into the summation rule and separating into two scalar equations gives

$$\frac{\sqrt{e_2}}{2} \omega_B - S_\theta \omega_S = 0$$

$$-\frac{\sqrt{e_2}}{2} \omega_B + C_\theta \omega_S = \frac{\sqrt{e_2}}{2} \left[ \frac{r(1+S_\theta+C_\theta)}{(b-rC_\theta)} \right] \omega_B$$

Multiplying the first equation by $C_\theta$ and the second by $S_\theta$, adding the two equations, and simplifying gives the final result.

$$b = \frac{r(1+S_\theta)}{C_\theta - S_\theta}$$

Rolling (without Slipping) in Two Dimensions – Point Contact

This section discusses an idealized model of a portion of a planetary gear system. The surface $S$ represents the sun gear, and the disk $D$ represents a planet gear. The circular surfaces of $S$ and $D$ represent the pitch circles of these gears.

Rolling on a Fixed Surface

If a rigid body rolls (without slipping) on a fixed surface, the point that is in contact with the surface has zero velocity. For example, consider a circular disk $D$ that rolls on the fixed circular surface $S$ as shown. Because $C'$ is in contact with the point $C$ on the fixed surface, its velocity is zero.

Using this result, the velocity of $G$ the center of the disk can be calculated using the relative velocity equation.

$$\frac{\dot{y}_G}{\dot{y}_C^{\text{zero}}} + \frac{\dot{y}_{G/C}}{\dot{y}_{C/C}} = \frac{\dot{y}_{G/C}}{\dot{y}_{C/C}} = \frac{\dot{e}_D \times \varepsilon_{G/C}}{\omega \times (-r \varepsilon_n)} \Rightarrow \frac{\dot{y}_G}{\dot{y}_C^{\text{zero}}} = v \varepsilon_t = r \omega \varepsilon_t$$

The acceleration of $G$ is calculated by differentiating the expression for $\frac{\dot{y}_G}{\dot{y}_C^{\text{zero}}}$.
\[
\frac{\mathbf{a}_G}{dt} = \frac{R}{v} \mathbf{e}_t = \mathbf{e}_t \cdot (R \mathbf{\dot{\omega}} + r \mathbf{\omega} \times \mathbf{e}_t) = \frac{v}{R + r} \mathbf{e}_n
\]

This last result is expected because the tangential acceleration of \(G\) is the rate of change of its speed and the normal acceleration is its velocity squared divided by the radius of curvature \(v^2/\rho = v^2/(R + r)\).

### Rolling on a Moving Surface

If a rigid body rolls (without slipping) on a moving surface, then the velocities of the two contact points \(C\) and \(C'\) must be the same (i.e. \(v_{C'} = v_C\)). For example, consider a circular disk \(D\) that rolls on the rotating circular surface \(S\) as shown in the figure. As before, the velocity of the mass center \(G\) can be calculated using the relative velocity equation.

\[
\mathbf{v}_G = \mathbf{v}_C + \mathbf{v}_{G/C} = \mathbf{v}_S \times \mathbf{e}_{C/O} + \mathbf{v}_D \times \mathbf{e}_{G/C} = \Omega \mathbf{\hat{k}} \times (-R \mathbf{e}_n) + \omega \mathbf{\hat{k}} \times (-R \mathbf{e}_n)
\]

\[
\Rightarrow \mathbf{v}_G = v \mathbf{e}_t = (R \Omega + r \omega) \mathbf{e}_t
\]

Again, the acceleration of \(G\) is found by differentiating the velocity vector.

\[
\frac{\mathbf{a}_G}{dt} = \frac{R}{(R \Omega + r \omega)} \mathbf{e}_t = (R \mathbf{\ddot{\Omega}} + r \mathbf{\dot{\omega}}) \mathbf{e}_t + (R \Omega + r \omega) \mathbf{\dot{e}}_t
\]

\[
= (R \mathbf{\ddot{\Omega}} + r \mathbf{\dot{\omega}}) \mathbf{e}_t + (R \Omega + r \omega) (\mathbf{\dot{\mathbf{\hat{k}}} \times \mathbf{e}_t})
\]

\[
= (R \mathbf{\ddot{\Omega}} + r \mathbf{\dot{\omega}}) \mathbf{e}_t + (R \Omega + r \omega) \left(\frac{v}{R + r} \mathbf{e}_n\right) \Rightarrow \mathbf{a}_G = (R \mathbf{\ddot{\Omega}} + r \mathbf{\dot{\omega}}) \mathbf{e}_t + \left(\frac{(R \Omega + r \omega)^2}{R + r}\right) \mathbf{e}_n
\]

Again, this last result is expected because the tangential acceleration of \(G\) is the rate of change of its speed and the normal acceleration is its velocity squared divided by the radius of curvature \(v^2/\rho = v^2/(R + r)\).

### Notes:

- Even though the velocities of the contact points are equal (i.e. \(v_{C'} = v_C\)), the accelerations of these points are not (i.e. \(a_{C'} \neq a_C\) and \(a_{C'} \neq a_C\)). However, for two dimensional systems, it can be shown that the components of these accelerations along the tangential direction \(\mathbf{e}_t\) are equal (i.e. \(a_{C'} \cdot \mathbf{e}_t = a_C \cdot \mathbf{e}_t\)).
In practice, planetary gear systems have a **separate component** that keeps the **centers** of its planetary gears at a **fixed distance** from the center of the sun gear so the planetary gears will **remain in contact** with the sun gear at all times.

Planetary gear systems may also have a ring gear that circumscribes the planetary gears. In this case, the sun and planetary gears are external gears (teeth on the outer surface of the gear), the ring gear is an internal gear (teeth on the inner surface of the gear). The ring gear may be stationary or it may also rotate.

**Rolling (without slipping) – Line Contact**

Cone Rolling on a Flat Plane

Consider a **right circular cone** \( C \) rolling without slipping on a **flat plane**. If the plane is **fixed**, then all the points on the line of contact of the cone must have **zero velocity**, and the cone will roll in a **circular path** with the point \( O \) remaining fixed.

To analyze the kinematics of the cone as it rolls, first define the reference frame \( S : (n_1, n_2, k) \). The unit vectors of \( S \) are directed as follows: \( n_1 \) is directed **along the line of contact**, \( k \) is **normal** to the plane, and \( n_2 = k \times n_1 \). Because all the points on the contact line have zero velocity, the **angular velocity** of the cone must be along the \( n_1 \) direction, that is, \[ R\omega_C = \omega n_1 \].

The **velocity** of any point of the cone may be calculated using the **relative velocity** equation. For example, suppose \( P C \) is a point on the contact line, and \( P \) is a point along the centerline of the cone, a distance \( a \) directly above \( P C \) (distance \( a \) is measured **normal** to the plane). Then, the velocity of \( P \) may be calculated as

\[
R\psi_P = R\psi_{PC} + \left( R\omega_C \times R\psi_{PC} \right) = \omega n_1 \times a k \quad \Rightarrow \quad R\psi_P = -a \omega n_2
\]

Now, consider a set of points \( P \) from the vertex at \( O \) to the center of the cone’s base at \( Q \). As point \( P \) is moved from \( O \) to \( Q \), the length \( a \) (and, hence, the velocity of the point \( P \)) **increases linearly**. Consequently, the cone rolls in a **circular path** with the point \( O \) remaining **fixed**.

The **angular acceleration** of the cone is found by **differentiating** \( R\omega_C \).

\[
R\alpha_C = \frac{d}{dt} \left( \omega n_1 \right) = \dot{\omega} n_1 + \omega \dot{n}_1 + \omega \left( R\omega_S \times n_1 \right) = \dot{\omega} n_1 + \omega \left( \Omega k \times n_1 \right) \quad \Rightarrow \quad R\alpha_C = \dot{\omega} n_1 + \omega \Omega n_2
\]

Here, \( R\omega_S = \Omega k \) represents the angular velocity of the frame \( S \) relative to the fixed plane. Hence, it describes the **rotation rate of the line of contact** within the plane.
Because the cone is rolling (without slipping), the angular rates $\omega$ and $\Omega$ are not independent. To find a relationship between the two rates, consider again a point $P$ on the centerline of the cone. As noted above, the velocity of $P$ may be written as $\mathbf{v}_P = -a \omega \mathbf{n}_2$. However, it can also be written as $\mathbf{v}_P = b \Omega \mathbf{n}_2$ where $b$ represents the distance from $P_C$ to $O$. Comparing these two expressions yields the result, $\Omega = -(a/b) \omega = -\omega \tan(\beta)$, where $\beta$ represents the half angle of the cone. Using this result, $\mathbf{a}_C$ the angular acceleration of the cone may be written as $\mathbf{a}_C = \dot{\omega} \mathbf{n}_1 - \omega^2 \tan(\beta) \mathbf{n}_2$. The acceleration of $P$ is found by differentiating the velocity $\mathbf{v}_P$.

$$\mathbf{a}_p = \frac{d}{dt}(-a \omega \mathbf{n}_2) = -a(\dot{\omega} \mathbf{n}_2 + \omega \dot{\mathbf{n}}_2) = -a\left(\dot{\omega} \mathbf{n}_2 + \omega (\Omega \mathbf{k} \times \mathbf{n}_2)\right) = -a\left(\dot{\omega} \mathbf{n}_2 - \omega \Omega \mathbf{n}_1\right)$$

or

$$\mathbf{a}_p = -a \dot{\omega} \mathbf{n}_2 - a \omega^2 \tan(\beta) \mathbf{n}_1$$

Beveled Gears

Beveled gears are a practical example of bodies that roll with a line of contact. Each gear may be thought of as a truncated cone, so two contacting gears may be thought of as two cones rolling on each other. Consider, for example, the system of two beveled gears shown. Gear $B$ is keyed to a vertical shaft that rotates with speed $\Omega$ about the $\mathbf{k}$ direction. Gear $A$ rolls on gear $B$ and rotates freely on the axle $S$. The axle rotates at a rate $\omega_2$ about the $\mathbf{k}$ direction (pivoting about point $O$), and gear $A$ rotates relative to the axle at a rate $\omega_{A/S}$.

To analyze the kinematics of this system, first define a unit vector set fixed in the axle as $S: (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$. The direction $\mathbf{n}_3$ is pointed along the axle towards $O$, the direction $\mathbf{n}_2$ is perpendicular to the axle as shown in the diagram, and the direction $\mathbf{n}_1 = \mathbf{n}_2 \times \mathbf{n}_3$. The angles $\alpha$ and $\beta$ are the half angles of the two cones (as shown), and the sum of these angles is defined here as angle $\gamma = \alpha + \beta$. Point $C$ is on the contact line between the two gears, and point $P$ is a point on the centerline of axle $S$.

The concept of relative velocity can be used to find a relationship between the angular rates $\Omega$, $\omega_2$, and $\omega_{A/S}$. If point $C$ is the contact point on gear $A$, the velocity $\mathbf{v}_C$ may be calculated as follows.
\[
\dot{\mathbf{v}}_C = \dot{\mathbf{v}}_P + \dot{\mathbf{v}}_{C/P}
\]

where
\[
\dot{\mathbf{v}}_P = -(b + aC_y)\omega_z n_1 \ldots (P \text{ moves in a circle around the vertical axis with radius } r = b + aC_y)
\]
\[
\dot{\mathbf{v}}_{C/P} = \mathbf{r}_A \times \dot{\mathbf{r}}_{C/P} = (\mathbf{r}_A + \mathbf{s}_A) \times (-a n_2) = a(\omega_{AIS} + \omega_z C_y) n_2
\]

Considering C to be the contact point on gear B, the velocity of C may also be written as \( \dot{\mathbf{v}}_C = -b\Omega n_1 \).

**Equating** the velocities of the two contact points and simplifying gives

\[
\omega_{AIS} = \left( \frac{b}{a} \right) (\omega_z - \Omega)
\]

**Check on the Relative Angular Velocities:** \( \dot{\omega}_A \) and \( \dot{\omega}_B \)

As noted above, when a cone rolls on a fixed plane, the angular velocity of the cone relative to the plane is directed along the line of contact. Similarly, when two beveled gears roll on each other, the relative angular velocities \( \dot{\omega}_A \) and \( \dot{\omega}_B \) must be directed along the line of contact. In the diagram, the unit vector \( \mathbf{m}_\perp \) is directed along the line of contact, and the unit vector \( \mathbf{m} \perp = n_1 \times \mathbf{m} \) is directed perpendicular to the line of contact.

\[
\dot{\omega}_A = \omega_{AIB} \mathbf{m} = -\dot{\omega}_B
\]

This result can be verified by computing the angular velocities and using the results found above. For example, using the angular velocity summation rule, \( \dot{\omega}_A \) may be calculated as follows

\[
\dot{\omega}_A = \dot{\omega}_A - \dot{\omega}_B = (\omega_z k + \omega_{AIS} n_3) - \Omega k = (\omega_z - \Omega) k + \omega_{AIS} n_3
\]
\[
= (\omega_z - \Omega) (C_\beta m + S_\beta m_\perp) + \frac{b}{a} (\omega_z - \Omega) (C_\alpha m - S_\alpha m_\perp)
\]
\[
= (\omega_z - \Omega) (C_\beta + \frac{b}{a} C_\alpha) m + (S_\beta - \frac{b}{a} S_\alpha) m_\perp
\]

where

\[
\frac{b}{a} = \frac{\mathbf{OC}}{\mathbf{a}}, \quad \frac{S_\beta}{S_\alpha} = \frac{\mathbf{OC}}{\mathbf{a}} \quad C_\beta + \frac{b}{a} C_\alpha = C_\beta + \frac{S_\beta C_\alpha}{S_\alpha} = \frac{C_\beta S_\alpha + S_\beta C_\alpha}{S_\alpha} = \frac{S_{a+b}}{S_\alpha}
\]
\[
S_\beta - \frac{b}{a} S_\alpha = S_\beta - \frac{S_\beta S_\alpha}{S_\alpha} = \frac{S_\beta S_\alpha - S_\beta S_\alpha}{S_\alpha} = 0
\]

**Substituting** these results into the above equation gives the final result showing that the relative angular velocities of the two gears are along the line of contact between them.

\[
\dot{\omega}_A = (\omega_z - \Omega) \left( \frac{S_{a+b}}{S_\alpha} \right) m
\]
Exercises

9.1 The sphere rolls without slipping on the flat plane. The Z-axis is normal to the plane, C is the point on the sphere in contact with the plane, and G is the center of the sphere. The angular velocity and angular acceleration of the sphere are given as

\[ \omega_S = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \quad \text{and} \quad \alpha_S = \dot{\omega}_x \mathbf{i} + \dot{\omega}_y \mathbf{j} + \dot{\omega}_z \mathbf{k} \]

Find \( \omega_G \) the velocity of G, \( \alpha_G \) the acceleration of G, and \( \alpha_C \) the acceleration of C.

Answers:

\[ \omega_G = r(\omega_y \mathbf{j} - \omega_x \mathbf{i}) \quad \alpha_G = r(\dot{\omega}_y \mathbf{j} - \dot{\omega}_x \mathbf{i}) \]

\[ \alpha_C = (-r \omega_x \omega_z) \mathbf{i} + (-r \omega_y \omega_z) \mathbf{j} + r(\omega_x^2 + \omega_y^2) \mathbf{k} \]

9.2 The system shown consists of a vertical column, a horizontal axle, and a wheel of radius \( r \). The horizontal arm rotates at a constant rate \( \Omega \), and the wheel \( W \) rolls without slipping in a circular arc. Find \( \omega_w \) and \( \alpha_w \) the angular velocity and angular acceleration of the wheel relative to a fixed frame, and find \( \omega_A \) and \( \alpha_A \) the velocity and acceleration of point A.

Answers:

\[ \omega_w = \Omega \mathbf{j} - (R / r) \Omega \mathbf{k} \quad \alpha_w = -(R / r) \Omega^2 \mathbf{i} \]

\[ \omega_A = R \Omega \mathbf{i} - R \Omega \mathbf{j} - r \Omega \mathbf{k} \quad \alpha_A = -\left( \frac{R^2 + r^2}{r} \right) \Omega^2 \mathbf{i} - R \Omega^2 \mathbf{k} \]

9.3 In the system shown, beveled gear A rolls on beveled gear B. As A rolls on B it spins about the axle OP which is pinned to the vertical shaft OE at O. If OE rotates at a constant angular velocity \( \omega_z \) (rad/sec), find \( \omega_A \) the angular velocity of gear A relative to the axle OP, \( \alpha_A \) the angular acceleration of gear A, and \( \alpha_C \) the acceleration of the contact point C of gear A.

Answers:

\[ \omega_A = \left( \frac{b}{a} \right) \omega_z \mathbf{n}_3 \]

\[ \alpha_A = (\frac{b}{a}) (\omega_z^2 S_\beta \mathbf{n}_2 + \omega_z (\frac{b}{a} + C_\beta) \mathbf{n}_3) \]

\[ \alpha_C = \omega_z^2 \left( (\frac{b}{a})(b + aC_\beta) \mathbf{n}_2 - b S_\beta \mathbf{n}_3 \right) \]
9.4 The figure depicts a planetary gear system moving in a fixed frame $R$. The ring gear $G$ has radius $r_G$, angular velocity $\dot{R} \omega_G$, and angular acceleration $\ddot{R} \alpha_G$. The sun gear $S$ has radius $r_S$, angular velocity $\dot{R} \omega_S$, and angular acceleration $\ddot{R} \alpha_S$. Show that $\dot{R} \omega_p$ the angular velocity of the planet gear and $\ddot{R} \alpha_p$ the angular acceleration of the planet gear may be written as

$$\dot{R} \omega_p = \left( r_G \dot{R} \omega_G + r_S \dot{R} \omega_S \right) / 2r_p$$

$$\ddot{R} \alpha_p = \left( r_G \ddot{R} \alpha_G + r_S \ddot{R} \alpha_S \right) / 2r_p$$

References: