An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume II: Kinetics

Unit 8

Basic Concepts of Linearization, Stability, Mode Shapes, and Natural Frequencies

Summary

The equations of motion of rigid body systems can be ***linear*** or ***nonlinear***. If the equations are ***nonlinear***, it may be possible to ***linearize*** them about some ***steady-state*** (equilibrium) positions. This unit describes how to find equilibrium positions (if they exist) and how to linearize the equations of motion about those positions. It also shows how to use the linear, approximate equations of motion to determine the ***stability*** of small motions about the equilibrium position. Finally, for stable equilibrium positions, this unit shows how to calculate the ***undamped natural frequencies*** and ***mode shapes*** associated with those positions. Examples are given for one and two degree-of-freedom systems. Extension to multi-degree-of-freedom systems is straightforward. MATLAB® is used to calculate eigenvalues and eigenvectors associated with the analysis.

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Linearization of Differential Equations of Motion

The equations of motion of a system can be ***linear*** or ***nonlinear***. If the equations are nonlinear, it may be possible to ***linearize*** them about some ***steady-state*** (equilibrium) position. In this unit, the ***linear approximate equations of motion*** are used to determine the ***stability*** of small motions about the steady-state position. They are also used to find the ***undamped natural frequencies*** and the ***mode shapes*** of some two-dimensional systems. These provide the frequencies and describe the types of small motions the system exhibits about these positions. Steady-state positions can be found from equations of statics, the principle of virtual work, or directly from the nonlinear equations of motion.

<u>Linearization of Functions of a Single Variable</u>

A ***nonlinear function*** $y = f(x)$ can be expanded in a ***Taylor series*** around an equilibrium position (or state), say x_{eq}, as follows.

$$f\left(x_{eq}+\Delta x\right) = f\left(x_{eq}\right) + \Delta x\left[\frac{df}{dx}\right]_{x=x_{eq}} + \frac{(\Delta x)^2}{2}\left[\frac{d^2 f}{dx^2}\right]_{x=x_{eq}} + \cdots \approx f\left(x_{eq}\right) + \Delta x\left[\frac{df}{dx}\right]_{x=x_{eq}}$$

Here Δx represents an ***excursion*** from the ***equilibrium state***. If the excursions are ***small***, then the approximation as stated in the second equation can be used to estimate changes in the function. That is,

$$\boxed{\Delta f\left(x\right) = f\left(x_{eq}+\Delta x\right) - f\left(x_{eq}\right) = m\,\Delta x} \qquad (1)$$

Here, $\Delta f\left(x\right)$ represents ***changes*** in the function $f(x)$, Δx represents ***changes*** in the system state away from the equilibrium, and the multiplier m is defined as follows.

$$\boxed{m = \left.\frac{df}{dx}\right|_{x=x_{eq}}} \qquad (2)$$

Eq. (1) represents a ***linear*** relationship between ***changes*** in the function f and ***changes*** in the state x.

<u>Linearization of Functions of Many Variables</u>

A ***nonlinear function*** $y = f(x_1, x_2, \ldots, x_n) = f\left(\underset{\sim}{x}\right)$, can be expanded in a ***Taylor series*** around the ***equilibrium state***, say $\underset{\sim}{x}_{eq} = \left[(x_1)_{eq}, (x_2)_{eq}, \ldots, (x_n)_{eq}\right]$, as follows.

$$f\left(\underset{\sim}{x}_{eq}+\Delta\underset{\sim}{x}\right) = f\left(\underset{\sim}{x}_{eq}\right) + \sum_{i=1}^{n}\Delta x_i\left[\frac{\partial f}{\partial x_i}\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}} + \cdots \approx f\left(\underset{\sim}{x}_{eq}\right) + \sum_{i=1}^{n}\Delta x_i\left[\frac{\partial f}{\partial x_i}\right]_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}}$$

Here the vector $\Delta\underset{\sim}{x} = \left(\Delta x_1, \Delta x_2, \ldots, \Delta x_n\right)$ represents an ***excursion*** from the equilibrium state. As before, if the excursions are ***small***, then the approximation as stated in the second equation can be used to estimate changes in the function. That is,

$$\Delta f(\underset{\sim}{x}) = f(\underset{\sim}{x}_{eq} + \Delta \underset{\sim}{x}) - f(\underset{\sim}{x}_{eq}) = \sum_{i=1}^{n} m_i \, \Delta x_i \tag{3}$$

Here, $\Delta f(\underset{\sim}{x})$ represents changes in the vector function $f(\underset{\sim}{x})$, $\Delta \underset{\sim}{x}$ represents changes in the system state $\underset{\sim}{x}$ away from the equilibrium, and the multipliers $m_i \; (i=1,\ldots,n)$ are defined as follows.

$$m_i = \left. \frac{\partial f}{\partial x_i} \right|_{\underset{\sim}{x}=\underset{\sim}{x}_{eq}} \qquad (i=1,\ldots,n) \tag{4}$$

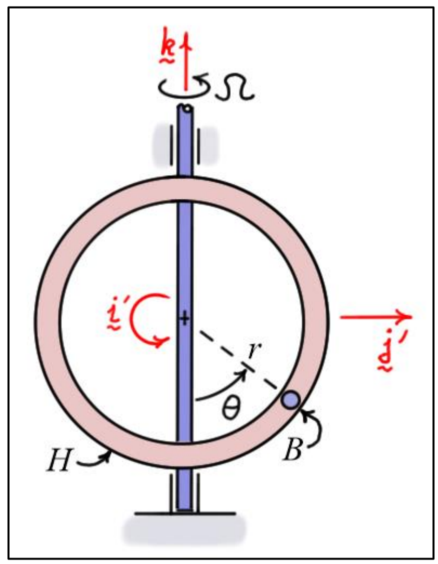
Eq. (3) represents a ***linear*** relationship between ***changes*** in the function f and ***changes*** in the ***elements*** of the state vector $\underset{\sim}{x}$. In this case, the function $f(x)$ is approximated by the sum of n linear terms.

Linearization of Ordinary Differential Equations

The above concepts can be used to ***linearize ordinary differential equations of motion***. The ***left sides*** of these equations generally are ***sums*** of ***terms*** involving the ***state variables*** and ***their derivatives***, and the ***right sides*** are ***sums*** of ***terms*** involving the ***forcing*** (or ***excitation***) ***functions***. To ***linearize*** the equations of motion, ***each*** of the ***terms*** (on the left side) involving the state variables must be ***linearized*** about the ***steady-state position***. To understand how this is done, consider the following examples.

Example 1:

The figure shows a ***hoop*** *H* of radius r rotating at a ***constant rate*** Ω (rad/s) about a ***vertical axis*** (annotated by the ***unit vector*** $\underset{\sim}{k}$). A ***bead*** *B* of mass m moves within the hoop as it rotates. The bead's ***position*** relative to *H* is defined by the angle θ. The following differential equation of motion describes the ***damped*** motion of *B* within the hoop.



$$\ddot{\theta} + (c/m)\dot{\theta} - \Omega^2 \sin(\theta)\cos(\theta) + (g/r)\sin(\theta) = 0 \tag{5}$$

Here, c represents the ***damping coefficient*** and g represents the ***acceleration of gravity***.

Find:

1) The ***equilibrium positions*** of *B* for constant Ω.
2) An ***approximate linear equation of motion*** about one of those positions.

Solution:

1) The equilibrium positions of *B* can be found from the differential equation of motion by setting $\dot{\theta} = \ddot{\theta} = 0$. That is, set

$$\frac{g}{r}\sin(\theta) - \Omega^2 \sin(\theta)\cos(\theta) = 0 \quad \Rightarrow \quad \left(\frac{g}{r} - \Omega^2 \cos(\theta)\right)\sin(\theta) = 0 \tag{6}$$

Eq. (6) is true under the following ***two conditions***.

a) $\boxed{\sin(\theta)=0}$ $\Rightarrow$ $\boxed{\theta=\begin{cases}0\\ \pi\end{cases}}$ and b) $\boxed{\frac{g}{r}-\ \Omega^2\cos(\theta)\ =\ 0}$ $\Rightarrow$ $\boxed{\theta=\cos^{-1}\left(\dfrac{g}{r\,\Omega^2}\right)}$

These conditions give ***three*** equilibrium (or steady-state) positions of *B*. The first two positions (from condition (a)) are at the bottom and top of *H*, and the third (from condition (b)) represents a position between zero and ninety degrees. Note, however, that the third position exists and is different from the first position (i.e. $\theta=0$) only if $\Omega^2>\frac{g}{r}$.

2) To illustrate the linearization process, the equation of motion is now ***linearized*** about $\theta=0$. The left side of the equation of motion has four terms. The ***first two*** are ***linear*** and the ***latter two*** are ***nonlinear***. To linearize the equation, the third and fourth terms must be linearized.

third term: $f(\theta)=\Omega^2\sin(\theta)\cos(\theta)$ $\Rightarrow$ $f'(\theta)\big|_{\theta=0}=\Omega^2\left(\cos^2(\theta)-\sin^2(\theta)\right)_{\theta=0}=\Omega^2$

$\Rightarrow$ linear approximation: $\boxed{\Delta f=\left(f'(\theta)\big|_{\theta=0}\right)\Delta\theta=\Omega^2\Delta\theta}$

fourth term: $f(\theta)=(g/r)\sin(\theta)$ $\Rightarrow$ $f'(\theta)\big|_{\theta=0}=(g/r)\left(\cos(\theta)\right)_{\theta=0}=(g/r)$

$\Rightarrow$ linear approximation: $\boxed{\Delta f=\left(f'(\theta)\big|_{\theta=0}\right)\Delta\theta=(g/r)\Delta\theta}$

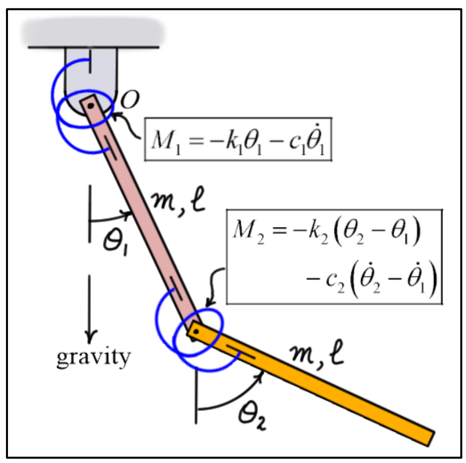
Substituting these results into the Eq. (5) gives the ***linear-approximate equation of motion***.

$$\boxed{\Delta\ddot{\theta}+(c/m)\Delta\dot{\theta}+\left(\tfrac{g}{r}-\Omega^2\right)\Delta\theta=0} \qquad (7)$$

Eq. (7) can be used to study ***small motions*** around $\theta=0$. A similar process can be followed to find approximate linear equations of motion about the other two positions.

Example 2:

The figure shows a ***double pendulum*** with ***rotational spring-dampers*** at its connecting joints. The spring-dampers are assumed to depend ***linearly*** on the ***relative angles*** and ***relative angular rates***. It can be shown that the ***equations of motion*** of the system under the action of gravity can be written as shown in Eqs. (8) and (9) below. The two equations represent a set of coupled ***second-order***, ***nonlinear***, ***ordinary differential equations***.



$$\boxed{\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1+\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2-\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2+\tfrac{3}{2}mg\ell S_1+\left(c_1+c_2\right)\dot{\theta}_1-c_2\dot{\theta}_2+\left(k_1+k_2\right)\theta_1-k_2\theta_2=0} \qquad (8)$$

$$\boxed{\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1+\left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2+\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2+\tfrac{1}{2}mg\ell S_2+c_2\left(\dot{\theta}_2-\dot{\theta}_1\right)+k_2\left(\theta_2-\theta_1\right)=0} \qquad (9)$$

Find:

1) The ***equilibrium position*** of the system

2) ***Approximate linear*** equations of motion about that position

Solution:

The equilibrium positions of the system are found from the differential equations by setting $\dot{\theta}_1 = \dot{\theta}_2 = \ddot{\theta}_1 = \ddot{\theta}_2 = 0$. That is, set

$$\boxed{\tfrac{3}{2} mg\ell S_1 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0} \quad \text{and} \quad \boxed{\tfrac{1}{2} mg\ell S_2 + k_2(\theta_2 - \theta_1) = 0} \qquad (10)$$

It is clear that $\boxed{\theta_1 = \theta_2 = 0}$ is a solution to Eqs. (10). Physically this is the condition where both bars are hanging ***vertically downward*** and the rotational springs at the joints are ***unstretched***.

To linearize Eqs. (8) and (9) about this position, each of the terms in the equations must be linearized. Consider, first, the terms in Eq. (8).

1st term: $\left(\tfrac{4}{3} m\ell^2\right)\ddot{\theta}_1$ … This term is a linear function of $\ddot{\theta}_1$.

2nd term: $\left(\tfrac{1}{2} m\ell^2 C_{2-1}\right)\ddot{\theta}_2$ … This term is a nonlinear function of θ_1, θ_2, and $\ddot{\theta}_2$.

Setting $f(\theta_1,\theta_2,\ddot{\theta}_2) = \tfrac{1}{2} m\ell^2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) \;\Rightarrow\; \boxed{\Delta f \approx m_1\Delta\theta_1 + m_2\Delta\theta_2 + m_3\Delta\ddot{\theta}_2}$

$$\boxed{m_1 = \left(\partial f/\partial\theta_1\right)\Big|_{eq} = \tfrac{1}{2} m\ell^2\left[\ddot{\theta}_2 \sin(\theta_2 - \theta_1)\right]_{eq} = 0} \quad \boxed{m_2 = \left(\partial f/\partial\theta_2\right)\Big|_{eq} = \tfrac{1}{2} m\ell^2\left[-\ddot{\theta}_2 \sin(\theta_2 - \theta_1)\right]_{eq} = 0}$$

$$\boxed{m_3 = \left(\partial f/\partial\ddot{\theta}_2\right)\Big|_{eq} = \tfrac{1}{2} m\ell^2\left[\cos(\theta_2 - \theta_1)\right]_{eq} = \tfrac{1}{2} m\ell^2}$$

$$\Rightarrow \boxed{\Delta f \approx \left(\tfrac{1}{2} m\ell^2\right)\Delta\ddot{\theta}_2}$$

3rd term: $\left(\tfrac{1}{2} m\ell^2 S_{2-1}\right)\dot{\theta}_2^2$ … This term is a nonlinear function of θ_1, θ_2, and $\dot{\theta}_2$.

Setting $f(\theta_1,\theta_2,\dot{\theta}_2) = \tfrac{1}{2} m\ell^2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \;\Rightarrow\; \boxed{\Delta f \approx m_1\Delta\theta_1 + m_2\Delta\theta_2 + m_3\Delta\dot{\theta}_2}$

$$\boxed{m_1 = \left(\partial f/\partial\theta_1\right)_{eq} = \tfrac{1}{2} m\ell^2\left[-\dot{\theta}_2^2 \cos(\theta_2 - \theta_1)\right]_{eq} = 0} \quad \boxed{m_2 = \left(\partial f/\partial\theta_2\right)_{eq} = \tfrac{1}{2} m\ell^2\left[\dot{\theta}_2^2 \cos(\theta_2 - \theta_1)\right]_{eq} = 0}$$

$$\boxed{m_3 = \left(\partial f/\partial\dot{\theta}_2\right)_{eq} = \tfrac{1}{2} m\ell^2\left[2\dot{\theta}_2 \sin(\theta_2 - \theta_1)\right]_{eq} = 0}$$

$$\Rightarrow \boxed{\Delta f \approx 0}$$

4th term: $\tfrac{3}{2} mg\ell S_1$ … This is a nonlinear function of θ_1.

Setting $f(\theta_1) = \tfrac{3}{2} mg\ell \sin(\theta_1)$

$$\Rightarrow \boxed{\Delta f \approx \left(\partial f/\partial\theta_1\right)_{eq} \Delta\theta_1 = \tfrac{3}{2} mg\ell\left(\cos(\theta_1)\right)_{eq} \Delta\theta_1 = \left(\tfrac{3}{2} mg\ell\right)\Delta\theta_1}$$

Carrying out a similar process for the second equation yields similar results. The two ***linear-approximate differential equations of motion*** are

$$\boxed{\left(\tfrac{4}{3}m\ell^2\right)\Delta\ddot{\theta}_1+\left(\tfrac{1}{2}m\ell^2\right)\Delta\ddot{\theta}_2+\left(c_1+c_2\right)\Delta\dot{\theta}_1-c_2\Delta\dot{\theta}_2+\left(k_1+k_2+\tfrac{3}{2}mg\ell\right)\Delta\theta_1-k_2\Delta\theta_2=0} \tag{11}$$

$$\boxed{\left(\tfrac{1}{2}m\ell^2\right)\Delta\ddot{\theta}_1+\left(\tfrac{1}{3}m\ell^2\right)\Delta\ddot{\theta}_2-c_2\Delta\dot{\theta}_1+c_2\Delta\dot{\theta}_2-k_2\Delta\theta_1+\left(k_2+\tfrac{1}{2}mg\ell\right)\Delta\theta_2=0} \tag{12}$$

Writing Eqs. (11) and (12) in ***matrix form*** gives

$$\boxed{[M]\begin{Bmatrix}\Delta\ddot{\theta}_1\\ \Delta\ddot{\theta}_2\end{Bmatrix}+[C]\begin{Bmatrix}\Delta\dot{\theta}_1\\ \Delta\dot{\theta}_2\end{Bmatrix}+[K]\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}} \tag{13}$$

Here,

$$\boxed{[M]\triangleq\begin{bmatrix}\tfrac{4}{3}m\ell^2 & \tfrac{1}{2}m\ell^2\\ \tfrac{1}{2}m\ell^2 & \tfrac{1}{3}m\ell^2\end{bmatrix}}\quad\boxed{[C]\triangleq\begin{bmatrix}c_1+c_2 & -c_2\\ -c_2 & c_2\end{bmatrix}}\quad\boxed{[K]\triangleq\begin{bmatrix}k_1+k_2+\tfrac{3}{2}mg\ell & -k_2\\ -k_2 & k_2+\tfrac{1}{2}mg\ell\end{bmatrix}} \tag{14}$$

These are called the system's ***mass***, ***damping***, and ***stiffness matrices***, respectively.

Linearization of Equations of Motion for Multibody Systems

<u>Nonlinear Equations of Motion</u>

The state vector of an "*n*" degree-of-freedom multibody system can be defined as the concatenation of a vector of ***generalized coordinates***, say $\{x\}_{n\times 1}$, and a vector of ***generalized speeds***, say $\{y\}_{n\times 1}$. For example, the two vectors could be defined as follows.

$$\boxed{\{x\}=\begin{Bmatrix}\{x_1\}\\ \{x_2\}\end{Bmatrix}\triangleq\begin{Bmatrix}\{\theta\}\\ \{s\}\end{Bmatrix}}\quad\boxed{\{y\}=\begin{Bmatrix}\{y_1\}\\ \{y_2\}\end{Bmatrix}\triangleq\begin{Bmatrix}\{\dot{\theta}\}\\ \{\dot{s}\}\end{Bmatrix}} \tag{15}$$

For simplicity, it is assumed here that $\{\theta\}$ represents the sets of ***orientation angles*** of all the bodies, and $\{s\}$ represents ***translational coordinates*** of various points within the system. The vector $\{y\}$ represents, the generalized speeds and are defined here to be the ***time-derivatives*** of the ***generalized coordinates***. The state vector of the system can then be written as

$$\boxed{\{z\}=\begin{Bmatrix}\{x\}\\ \{y\}\end{Bmatrix}} \tag{16}$$

Using any of a variety of approaches (Newton/Euler equations, Lagrange's equations, Kane's equations, etc.), the ***equations of motion*** of a ***multibody system*** can be written in the form

$$\boxed{[M]\{\dot{y}\}=\{f\}}\quad\Rightarrow\boxed{\{\dot{y}\}=[M]^{-1}\{f\}=\{g\}} \tag{17}$$

Here, $\left[M\right]$ is called the ***generalized mass matrix*** and is a function of $\{x\}$ the ***generalized coordinates***, and the right-side vector $\{f\}$ is a function of $\{x\}$ the ***generalized coordinates***, $\{y\}$ the ***generalized speeds***, and $\{u(t)\}$ a vector of ***external excitations*** (or external input). These functional dependencies are generally ***non-linear***.

To numerically solve the equations and simulate the motion over time, these equations are supplemented with a set of kinematical differential equations. In this case,

$$\boxed{\{\dot{x}\} = \{y\}} \qquad (18)$$

Eqs. (16)-(18) can be combined into a single set of equations for the system. That is,

$$\boxed{\{\dot{z}\} = \begin{Bmatrix} \{\dot{x}\} \\ \{\dot{y}\} \end{Bmatrix} = \begin{Bmatrix} \{y\} \\ \left[M\right]^{-1}\{f\} \end{Bmatrix} = \begin{Bmatrix} \{y\} \\ \{g\} \end{Bmatrix}} \qquad (19)$$

<u>Steady-State Equilibrium Configurations</u>

Under certain conditions, multibody systems can exhibit ***steady-state equilibrium positions***. In these positions, the values of the ***generalized coordinates*** are ***constant***. To find the values of the generalized coordinates associated with these configurations, first set the entries of the external input vector to their constant equilibrium values, that is, $\{u\}_{m\times1} = \{u_e\}_{m\times1}$. Then, assuming the generalized speeds are zero $\left(\text{i.e. } \{y\} \equiv \{0\}\right)$, set

$$\boxed{\{\dot{y}\} = \left\{g\left(\{x_e\},\{u_e\}\right)\right\} = 0} \qquad (20)$$

Eq. (20) represents a set of "*n*" ***nonlinear algebraic equations*** in the "*n*" ***unknown generalized coordinates*** $\{x_e\}$ that define the equilibrium configuration.

It should be noted here that as a set of nonlinear algebraic equations, Eq. (20) is ***not guaranteed*** to have a ***solution***. For example, if the external input vector $\{u_e\}$ is ***not consistent*** with an equilibrium condition of the system, then ***no solution will exist***. Moreover, if the external input is consistent with a ***weak equilibrium position***, a solution may exist, but it may be ***difficult*** to ***calculate numerically***. Nonlinear algebraic equations ***can be very difficult to solve*** even if a solution exists; however, the algorithms to solve these types of equations have improved immensely over the years.

<u>Linearization of Equations of Motion</u>

To describe the ***perturbed motion*** of the multibody system ***near*** an ***equilibrium position***, the equations of motion can be ***linearized*** about the equilibrium state. To this end, perturbations are defined for the system's generalized coordinates, generalized speeds, and the external input vector as follows.

$$\boxed{\{x\} = \{x_e\} + \{\Delta x\}} \quad \boxed{\{y\} = \underbrace{\{y_e\}}_{\text{zero}} + \{\Delta y\}} \quad \boxed{\{u\} = \{u_e\} + \{\Delta u\}} \qquad (21)$$

Substituting these ***perturbed values*** into the equations of motion, expanding in a ***Taylor Series*** about the ***equilibrium values***, and omitting terms of second and higher order in the perturbations gives.

$$\left.\begin{aligned} &\{\dot{x}\} = \{\Delta\dot{x}\} = \{y\} \\ &\{\dot{y}\} = \{\Delta\dot{y}\} = \Big\{ g\Big(\underbrace{\{x_e\} + \{\Delta x\}, \{y_e\} + \{\Delta y\}, \{u_e\} + \{\Delta u\}}_{\text{zero}} \Big)\Big\} \approx \left[A_1 \mid A_2 \right] \begin{Bmatrix} \Delta x \\ \Delta y \end{Bmatrix} + [B]\{\Delta u\} \end{aligned}\right\} \quad (22)$$

Here, the elements of the matrices $[A_1]$, $[A_2]$, and $[B]$ are defined as

$$\left[A_1\right]_{ij} \triangleq \left.\frac{\partial g_i}{\partial x_j}\right|_{eq} \quad (i=1,\ldots,n;\ j=1,\ldots,n) \qquad \left[A_2\right]_{ij} \triangleq \left.\frac{\partial g_i}{\partial y_j}\right|_{eq} \quad (i=1,\ldots,n;\ j=1,\ldots,n) \quad (23)$$

$$[B]_{ik} \triangleq \left.\frac{\partial g_i}{\partial u_k}\right|_{eq} \quad (i=1,\ldots,n;\ k=1,\ldots,m) \quad (24)$$

The partial derivatives of Eqs. (23) and (24) are ***evaluated*** at the ***equilibrium state***, hence the matrices $[A_1]$, $[A_2]$, and $[B]$ are ***constant matrices***.

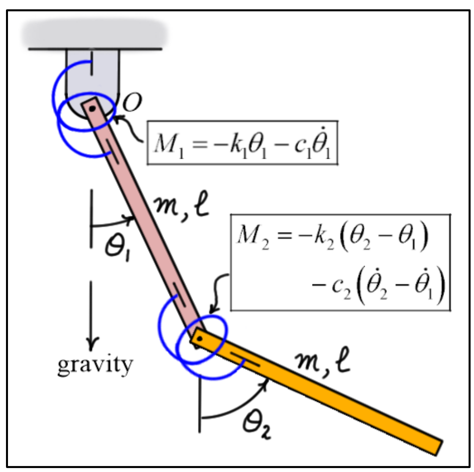
Eqs. (22) represent a set of ***first-order***, ***linear***, ***ordinary differential equations*** that describe the ***perturbed motion*** of the system about its equilibrium state. They can be combined into the following partitioned matrix form.

$$\{\dot{z}\} = \begin{Bmatrix} \{\Delta\dot{x}\} \\ \{\Delta\dot{y}\} \end{Bmatrix} = \left[\begin{array}{c|c} [\,0\,] & [\,I\,] \\ \hline [\,A_1\,] & [\,A_2\,] \end{array}\right] \begin{Bmatrix} \{\Delta x\} \\ \{\Delta y\} \end{Bmatrix} + \begin{bmatrix} [\,0\,] \\ [\,B\,] \end{bmatrix} \{\Delta u\} \quad (25)$$

Eqs. (25) are useful for studying ***stability*** of the perturbed motion, perturbation ***sensitivity***, and ***control system*** design.

Example 3:

As stated above the equations of motion of the double pendulum system shown here can be written as follows.



$$\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2}mg\ell S_1 + (c_1+c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0$$

$$\left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2}mg\ell S_2 + c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) + k_2\left(\theta_2 - \theta_1\right) = 0$$

Find:

Linearized equations of motion for this system about the equilibrium state $(\theta_1)_{eq} = (\theta_2)_{eq} = 0$.

Solution:

The two boxed equations above can be rewritten in the matrix form of Eq. (17) as shown below.

$$\begin{bmatrix} \left(\frac{4}{3}m\ell^2\right) & \left(\frac{1}{2}m\ell^2 C_{2-1}\right) \\ \left(\frac{1}{2}m\ell^2 C_{2-1}\right) & \left(\frac{1}{3}m\ell^2\right) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \frac{3}{2}mg\ell S_1 - \left(c_1+c_2\right)\dot{\theta}_1 + c_2\dot{\theta}_2 - \left(k_1+k_2\right)\theta_1 + k_2\theta_2 \\ -\left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \frac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k_2\left(\theta_2-\theta_1\right) \end{Bmatrix} \quad (26)$$

For this system, the vectors $\{x\}$, $\{y\}$, and $\{u\}$ can be defined as follows.

$$\{x\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \qquad \{y\} = \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} \qquad \{u\} = \{0\} \quad (27)$$

The second derivatives of the angles can be found by multiplying both sides of Eq. (26) by $[M]^{-1}$.

$$[M]^{-1} = \begin{bmatrix} \left(\frac{4}{3}m\ell^2\right) & \left(\frac{1}{2}m\ell^2 C_{2-1}\right) \\ \left(\frac{1}{2}m\ell^2 C_{2-1}\right) & \left(\frac{1}{3}m\ell^2\right) \end{bmatrix}^{-1} = \frac{1}{\frac{4}{9}m^2\ell^4 - \frac{1}{4}m^2\ell^4 C_{2-1}^2} \begin{bmatrix} \left(\frac{1}{3}m\ell^2\right) & -\left(\frac{1}{2}m\ell^2 C_{2-1}\right) \\ -\left(\frac{1}{2}m\ell^2 C_{2-1}\right) & \left(\frac{4}{3}m\ell^2\right) \end{bmatrix}$$

$$= \frac{m\ell^2}{\frac{1}{36}m^2\ell^4\left(16-9C_{2-1}^2\right)} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2}C_{2-1} \\ -\frac{1}{2}C_{2-1} & \frac{4}{3} \end{bmatrix}$$

$$= \frac{36}{m\ell^2\left(16-9C_{2-1}^2\right)} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2}C_{2-1} \\ -\frac{1}{2}C_{2-1} & \frac{4}{3} \end{bmatrix}$$

$$\Rightarrow \{\dot{y}\} = \frac{\left(36/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2}C_{2-1} \\ -\frac{1}{2}C_{2-1} & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} \left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \frac{3}{2}mg\ell S_1 - \left(c_1+c_2\right)\dot{\theta}_1 + c_2\dot{\theta}_2 - \left(k_1+k_2\right)\theta_1 + k_2\theta_2 \\ -\left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \frac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k_2\left(\theta_2-\theta_1\right) \end{Bmatrix}$$

The right side of the above equation defines the elements of the vector $\{g\}$. Carrying out the matrix multiplication gives

$$g_1 = \frac{\left(12/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)} \left[\left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \frac{3}{2}mg\ell S_1 - \left(c_1+c_2\right)\dot{\theta}_1 + c_2\dot{\theta}_2 - \left(k_1+k_2\right)\theta_1 + k_2\theta_2\right]$$
$$- \frac{\left(18/m\ell^2\right)C_{2-1}}{\left(16-9C_{2-1}^2\right)} \left[-\left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \frac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k_2\left(\theta_2-\theta_1\right)\right] \quad (28)$$

$$g_2 = \frac{\left(18/m\ell^2\right)C_{2-1}}{\left(16-9C_{2-1}^2\right)} \left[-\left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 + \frac{3}{2}mg\ell S_1 + \left(c_1+c_2\right)\dot{\theta}_1 - c_2\dot{\theta}_2 + \left(k_1+k_2\right)\theta_1 - k_2\theta_2\right]$$
$$- \frac{\left(48/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)} \left[\left(\frac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 + \frac{1}{2}mg\ell S_2 + c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) + k_2\left(\theta_2-\theta_1\right)\right] \quad (29)$$

To formulate the linearized equations of motion about the equilibrium state $\left(\theta_1\right)_{eq} = \left(\theta_2\right)_{eq} = 0$, the above expressions must be ***differentiated***. Specifically,

$$\left[A_1\right]_{ij} \triangleq \left.\frac{\partial g_i}{\partial x_j}\right|_{eq} \quad (i=1,2\,;\ j=1,2) \qquad \left[A_2\right]_{ij} \triangleq \left.\frac{\partial g_i}{\partial y_j}\right|_{eq} \quad (i=1,2\,;\ j=1,2)$$

Using the ***product*** and ***quotient*** rules, the partial derivatives of function g_1 can be calculated as follows.

$$\left.\frac{\partial g_1}{\partial \theta_1}\right|_{eq} = \left[\frac{\partial}{\partial \theta_1}\left[\frac{\left(12/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)}\left[\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell S_1 - \left(c_1+c_2\right)\dot{\theta}_1 + c_2\dot{\theta}_2 - \left(k_1+k_2\right)\theta_1 + k_2\theta_2\right]\right.\right.$$
$$\left.\left. - \frac{\left(18/m\ell^2\right)C_{2-1}}{\left(16-9C_{2-1}^2\right)}\left[-\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2-\dot{\theta}_1\right) - k_2\left(\theta_2-\theta_1\right)\right]\right]\right]_{eq}$$
$$= \left[\left(\frac{12}{m\ell^2}\right)\left(\frac{18 S_{2-1}C_{2-1}}{\left(16-9C_{2-1}^2\right)^2}\right)\left[\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell S_1 - \left(c_1+c_2\right)\dot{\theta}_1 + c_2\dot{\theta}_2 - \left(k_1+k_2\right)\theta_1 + k_2\theta_2\right]\right]_{eq}$$
$$+ \left[\frac{\left(12/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)}\left[\left(-\tfrac{1}{2}m\ell^2 C_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell C_1 - \left(k_1+k_2\right)\right]\right]_{eq}$$
$$- \left[\left(\frac{18}{m\ell^2}\right)\left(\frac{S_{2-1}\left(16+9C_{2-1}^2\right)}{\left(16-9C_{2-1}^2\right)^2}\right)\left[-\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2-\dot{\theta}_1\right) - k_2\left(\theta_2-\theta_1\right)\right]\right]_{eq}$$
$$- \left[\frac{\left(18/m\ell^2\right)C_{2-1}}{\left(16-9C_{2-1}^2\right)}\left[\left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\dot{\theta}_1^2 + k_2\right]\right]_{eq}$$

$$\Rightarrow \boxed{\left.\frac{\partial g_1}{\partial \theta_1}\right|_{eq} = -\left(\frac{18g}{7\ell} + \frac{12k_1}{7m\ell^2} + \frac{30k_2}{7m\ell^2}\right)}$$

$$\left.\frac{\partial g_1}{\partial \dot{\theta}_1}\right|_{eq} = \left[\frac{\partial}{\partial \dot{\theta}_1}\left[\frac{\left(12/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)}\left[\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell S_1 - \left(c_1+c_2\right)\dot{\theta}_1 + c_2\dot{\theta}_2 - \left(k_1+k_2\right)\theta_1 + k_2\theta_2\right]\right.\right.$$
$$\left.\left. - \frac{\left(18/m\ell^2\right)C_{2-1}}{\left(16-9C_{2-1}^2\right)}\left[-\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2-\dot{\theta}_1\right) - k_2\left(\theta_2-\theta_1\right)\right]\right]\right]_{eq}$$
$$= \left[\frac{\left(12/m\ell^2\right)}{\left(16-9C_{2-1}^2\right)}\left[-\left(c_1+c_2\right)\right] - \frac{\left(18/m\ell^2\right)C_{2-1}}{\left(16-9C_{2-1}^2\right)}\left[-\left(m\ell^2 S_{2-1}\right)\dot{\theta}_1 + c_2\right]\right]_{eq}$$
$$= -\frac{12}{7m\ell^2}\left(c_1+c_2\right) - \frac{18}{7m\ell^2}\left(c_2\right)$$

$$\Rightarrow \boxed{\left.\frac{\partial g_1}{\partial \dot{\theta}_1}\right|_{eq} = -\left(\frac{12c_1}{7m\ell^2} + \frac{30c_2}{7m\ell^2}\right)}$$

$$\left.\frac{\partial g_1}{\partial \theta_2}\right|_{eq} = \left[\frac{\partial}{\partial \theta_2}\left[\frac{(12/m\ell^2)}{(16-9C_{2-1}^2)}\left[\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell S_1 - (c_1+c_2)\dot{\theta}_1 + c_2\dot{\theta}_2 - (k_1+k_2)\theta_1 + k_2\theta_2\right]\right.\right.$$

$$\left.\left. - \frac{(18/m\ell^2)C_{2-1}}{(16-9C_{2-1}^2)}\left[-\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k_2\left(\theta_2 - \theta_1\right)\right]\right]\right]_{eq}$$

$$= \left[\left(\frac{12}{m\ell^2}\right)\left(\frac{-18S_{2-1}C_{2-1}}{(16-9C_{2-1}^2)^2}\right)\left[\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell S_1 - (c_1+c_2)\dot{\theta}_1 + c_2\dot{\theta}_2 - (k_1+k_2)\theta_1 + k_2\theta_2\right]\right]_{eq}$$

$$+\left[\frac{(12/m\ell^2)}{(16-9C_{2-1}^2)}\left[\left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\dot{\theta}_2^2 + k_2\right]\right]_{eq}$$

$$-\left[\left(\frac{18}{m\ell^2}\right)\left(\frac{-S_{2-1}(16+9C_{2-1}^2)}{(16-9C_{2-1}^2)^2}\right)\left[-\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k_2\left(\theta_2 - \theta_1\right)\right]\right]_{eq}$$

$$-\left[\frac{(18/m\ell^2)C_{2-1}}{(16-9C_{2-1}^2)}\left[-\left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell C_2 - k_2\right]\right]_{eq}$$

$$\Rightarrow \boxed{\left.\frac{\partial g_1}{\partial \theta_2}\right|_{eq} = \frac{9g}{7\ell} + \frac{30k_2}{7m\ell^2}}$$

$$\left.\frac{\partial g_1}{\partial \dot{\theta}_2}\right|_{eq} = \left[\frac{\partial}{\partial \dot{\theta}_2}\left[\frac{(12/m\ell^2)}{(16-9C_{2-1}^2)}\left[\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 - \tfrac{3}{2}mg\ell S_1 - (c_1+c_2)\dot{\theta}_1 + c_2\dot{\theta}_2 - (k_1+k_2)\theta_1 + k_2\theta_2\right]\right.\right.$$

$$\left.\left. - \frac{(18/m\ell^2)C_{2-1}}{(16-9C_{2-1}^2)}\left[-\left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 - \tfrac{1}{2}mg\ell S_2 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k_2\left(\theta_2 - \theta_1\right)\right]\right]\right]_{eq}$$

$$= \left[\frac{(12/m\ell^2)}{(16-9C_{2-1}^2)}\left[\left(m\ell^2 S_{2-1}\right)\dot{\theta}_2 + c_2\right] - \frac{(18/m\ell^2)C_{2-1}}{(16-9C_{2-1}^2)}\left[-c_2\right]\right]_{eq}$$

$$= \frac{12}{7m\ell^2}(c_2) + \frac{18}{7m\ell^2}(c_2)$$

$$\Rightarrow \boxed{\left.\frac{\partial g_1}{\partial \dot{\theta}_2}\right|_{eq} = \frac{30c_2}{7m\ell^2}}$$

Similarly, it can be shown that the partial derivatives of the function g_2 are as follows.

$$\boxed{\left.\frac{\partial g_2}{\partial \theta_1}\right|_{eq} = \frac{27g}{7\ell} + \frac{18k_1}{7m\ell^2} + \frac{66k_2}{7m\ell^2}} \quad \boxed{\left.\frac{\partial g_2}{\partial \theta_2}\right|_{eq} = -\left(\frac{24g}{7\ell} + \frac{66k_2}{7m\ell^2}\right)} \quad \boxed{\left.\frac{\partial g_2}{\partial \dot{\theta}_1}\right|_{eq} = \frac{18c_1}{7m\ell^2} + \frac{66c_2}{7m\ell^2}} \quad \boxed{\left.\frac{\partial g_2}{\partial \dot{\theta}_2}\right|_{eq} = -\frac{66c_2}{7m\ell^2}}$$

The above results can now be used to define the individual submatrices in Eq. (25). For this system, the submatrices are as follows.

$$\left[A_1\right]=\begin{bmatrix} -\left(\frac{18g}{7\ell}+\frac{12k_1}{7m\ell^2}+\frac{30k_2}{7m\ell^2}\right) & \left(\frac{9g}{7\ell}+\frac{30k_2}{7m\ell^2}\right) \\ \left(\frac{27g}{7\ell}+\frac{18k_1}{7m\ell^2}+\frac{66k_2}{7m\ell^2}\right) & -\left(\frac{24g}{7\ell}+\frac{66k_2}{7m\ell^2}\right) \end{bmatrix} \qquad \left[A_2\right]=\frac{1}{7m\ell^2}\begin{bmatrix} -\left(12c_1+30c_2\right) & 30c_2 \\ \left(18c_1+66c_2\right) & -66c_2 \end{bmatrix} \quad (30)$$

$$\left[B\right]=\left[0\right]$$

Eq. (25) can then be written as

$$\{\dot{z}\}=\begin{Bmatrix} \begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix} \\ \begin{Bmatrix} \Delta\ddot{\theta}_1 \\ \Delta\ddot{\theta}_2 \end{Bmatrix} \end{Bmatrix}=\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -\left(\frac{18g}{7\ell}+\frac{12k_1}{7m\ell^2}+\frac{30k_2}{7m\ell^2}\right) & \left(\frac{9g}{7\ell}+\frac{30k_2}{7m\ell^2}\right) & -\left(\frac{12c_1}{7m\ell^2}+\frac{30c_2}{7m\ell^2}\right) & \frac{30c_2}{7m\ell^2} \\ \left(\frac{27g}{7\ell}+\frac{18k_1}{7m\ell^2}+\frac{66k_2}{7m\ell^2}\right) & -\left(\frac{24g}{7\ell}+\frac{66k_2}{7m\ell^2}\right) & \frac{18c_1}{7m\ell^2}+\frac{66c_2}{7m\ell^2} & -\frac{66c_2}{7m\ell^2} \end{array}\right]\begin{Bmatrix} \begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix} \\ \begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix} \end{Bmatrix}$$

Check:

The above results can now be compared with the results presented in Eqs. (13) and (14) of Example 2. The first two of the above equations are simply definitions of the state vector, and the second set of two equations should be the same as those presented in Eqs. (13) and (14). Specifically,

$$[M]\begin{Bmatrix} \Delta\ddot{\theta}_1 \\ \Delta\ddot{\theta}_2 \end{Bmatrix}+[C]\begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix}+[K]\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \Delta\ddot{\theta}_1 \\ \Delta\ddot{\theta}_2 \end{Bmatrix}=-[M]^{-1}\left([C]\begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix}+[K]\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix}\right)$$

Expanding this result gives

$$\begin{Bmatrix} \Delta\ddot{\theta}_1 \\ \Delta\ddot{\theta}_2 \end{Bmatrix}=\frac{m\ell^2}{\left(\frac{4}{9}-\frac{1}{4}\right)\left(m\ell^2\right)^2}\begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{4}{3} \end{bmatrix}\left(\begin{bmatrix} -\left(c_1+c_2\right) & c_2 \\ c_2 & -c_2 \end{bmatrix}\begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix}+\begin{bmatrix} -\left(k_1+k_2+\frac{3}{2}mg\ell\right) & k_2 \\ k_2 & -\left(k_2+\frac{1}{2}mg\ell\right) \end{bmatrix}\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix}\right)$$

$$=\frac{36}{7m\ell^2}\begin{bmatrix} -\frac{1}{3}\left(c_1+c_2\right)-\frac{1}{2}c_2 & \frac{1}{3}c_2+\frac{1}{2}c_2 \\ \frac{1}{2}\left(c_1+c_2\right)+\frac{4}{3}c_2 & -\frac{1}{2}c_2-\frac{4}{3}c_2 \end{bmatrix}\begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix}$$

$$+\frac{36}{7m\ell^2}\begin{bmatrix} -\frac{1}{3}\left(k_1+k_2+\frac{3}{2}mg\ell\right)-\frac{1}{2}k_2 & \frac{1}{3}k_2+\frac{1}{2}\left(k_2+\frac{1}{2}mg\ell\right) \\ \frac{1}{2}\left(k_1+k_2+\frac{3}{2}mg\ell\right)+\frac{4}{3}k_2 & -\frac{1}{2}k_2-\frac{4}{3}\left(k_2+\frac{1}{2}mg\ell\right) \end{bmatrix}\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix}$$

$$=\frac{36}{7m\ell^2}\begin{bmatrix} -\frac{1}{3}c_1-\frac{5}{6}c_2 & \frac{5}{6}c_2 \\ \frac{1}{2}c_1+\frac{11}{6}c_2 & -\frac{11}{6}c_2 \end{bmatrix}\begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix}+\frac{36}{7m\ell^2}\begin{bmatrix} -\left(\frac{1}{3}k_1+\frac{5}{6}k_2+\frac{1}{2}mg\ell\right) & \frac{5}{6}k_2+\frac{1}{4}mg\ell \\ \frac{1}{2}k_1+\frac{11}{6}k_2+\frac{3}{4}mg\ell & -\left(\frac{11}{6}k_2+\frac{2}{3}mg\ell\right) \end{bmatrix}\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix}$$

From the final line in the above equation, the matrices $\left[A_1\right]$ and $\left[A_2\right]$ are identified to be

$$\left[A_1\right]=\frac{36}{7m\ell^2}\begin{bmatrix}-\left(\frac{1}{3}k_1+\frac{5}{6}k_2+\frac{1}{2}mg\ell\right) & \frac{5}{6}k_2+\frac{1}{4}mg\ell \\ \frac{1}{2}k_1+\frac{11}{6}k_2+\frac{3}{4}mg\ell & -\left(\frac{11}{6}k_2+\frac{2}{3}mg\ell\right)\end{bmatrix}=\begin{bmatrix}-\left(\frac{18g}{7\ell}+\frac{12k_1}{7m\ell^2}+\frac{30k_2}{7m\ell^2}\right) & \frac{9g}{7\ell}+\frac{30k_2}{7m\ell^2} \\ \frac{27g}{7\ell}+\frac{18k_1}{7m\ell^2}+\frac{66k_2}{7m\ell^2} & -\left(\frac{24g}{7\ell}+\frac{66k_2}{7m\ell^2}\right)\end{bmatrix}$$

$$\left[A_2\right]=\frac{36}{7m\ell^2}\begin{bmatrix}-\frac{1}{3}c_1-\frac{5}{6}c_2 & \frac{5}{6}c_2 \\ \frac{1}{2}c_1+\frac{11}{6}c_2 & -\frac{11}{6}c_2\end{bmatrix}=\frac{1}{7m\ell^2}\begin{bmatrix}-\left(12c_1+30c_2\right) & 30c_2 \\ 18c_1+66c_2 & -66c_2\end{bmatrix}$$

These results ***match*** with those presented in Eq. (30).

<u>Systems with Many Degrees-of-Freedom</u>

As the number of degrees-of-freedom ***increases***, the calculations required to obtain the ***partial derivatives*** that populate the matrices $\left[A_1\right]$, $\left[A_2\right]$, and $\left[B\right]$ become ***unwieldy***. For these systems, it is more practical to use ***finite-difference approximations*** to estimate the partial derivatives. For example, using ***second-order***, ***central differences***, the entries of the matrices $\left[A_1\right]$, $\left[A_2\right]$, and $\left[B\right]$ can be approximated as follows. First define a set of "$n\times 1$" vectors of ***small increments*** in the entries of the state vector, and a set of "$m\times 1$" vectors of ***small increments*** in the entries of the external input vector.

$$\left\{dx\right\}_j=\Big[0,\ldots,0,\ \underbrace{dx_j}_{j^{\text{th}}\text{ entry}}\ ,0,\ldots,0\Big]^T \qquad \left\{dy\right\}_j=\Big[0,\ldots,0,\ \underbrace{dy_j}_{j^{\text{th}}\text{ entry}}\ ,0,\ldots,0\Big]^T \qquad \left(j=1,\ldots,n\right) \tag{31}$$

$$\left\{du\right\}_k=\Big[0,\ldots,0,\ \underbrace{du_k}_{k^{\text{th}}\text{ entry}}\ ,0,\ldots,0\Big]^T \qquad \left(k=1,\ldots,m\right) \tag{32}$$

Then, the elements of the $\left[A_1\right]$, $\left[A_2\right]$, and $\left[B\right]$ matrices can be ***estimated*** as follows.

$$\left[A_1\right]_{ij}\triangleq\left.\frac{\partial g_i}{\partial x_j}\right|_{eq}\approx\frac{g_i\left(\left\{x_e\right\}+\left\{dx\right\}_j,\left\{y_e\right\},\left\{u_e\right\}\right)-g_i\left(\left\{x_e\right\}-\left\{dx\right\}_j,\left\{y_e\right\},\left\{u_e\right\}\right)}{2\,dx_j} \qquad \left(i,\,j=1,\ldots,n\right) \tag{33}$$

$$\left[A_2\right]_{ij}\triangleq\left.\frac{\partial g_i}{\partial y_j}\right|_{eq}\approx\frac{g_i\left(\left\{x_e\right\},\left\{y_e\right\}+\left\{dy\right\}_j,\left\{u_e\right\}\right)-g_i\left(\left\{x_e\right\},\left\{y_e\right\}-\left\{dy\right\}_j,\left\{u_e\right\}\right)}{2\,dy_j} \qquad \left(i,\,j=1,\ldots,n\right) \tag{34}$$

$$\left[B\right]_{ik}\triangleq\left.\frac{\partial g_i}{\partial u_k}\right|_{eq}\approx\frac{g_i\left(\left\{x_e\right\},\left\{y_e\right\},\left\{u_e\right\}+\left\{du\right\}_k\right)-g_i\left(\left\{x_e\right\},\left\{y_e\right\},\left\{u_e\right\}-\left\{du\right\}_k\right)}{2\,du_k} \qquad \left(i=1,\ldots,n;\,k=1,\ldots,m\right) \tag{35}$$

It should be noted here that the functions $g_i\left(\{x\},\{y\},\{u\}\right)\ (i=1,\ldots,n)$ need ***not*** be ***determined analytically*** in this process. The ***finite-difference calculations*** require only the ***values*** of the functions near the equilibrium state. Those values can be calculated numerically using the equations of motion.

Stability of Steady-State Equilibria for Single Degree-of-Freedom Systems

The dynamics of ***single degree-of-freedom*** rigid body systems are governed by a ***single second-order***, ***nonlinear, ordinary differential equation of motion***. As discussed in earlier sections, that equation of motion can often be ***linearized*** about positions of steady-state equilibrium. It is assumed here that the resulting linear equation has ***constant coefficients***.

The ***stability*** associated with ***infinitesimal motions*** of the system about the equilibrium positions can be determined by examining the ***roots*** of the ***characteristic equation*** of the linearized equation of motion. The characteristic equation can be found, for example, by applying the ***Laplace transform*** to that equation. The types of ***infinitesimal motions*** fall into ***three*** basic categories depending on the type of roots as shown in the following table. The ***infinitesimal motion*** is ***asymptotically stable*** if the roots have ***negative real parts*** and ***unstable*** if the roots have ***positive real parts***. These types of motion are said to have ***significant behavior***. The motion is ***constant*** or ***bounded oscillatory*** motion if the ***real parts*** of the roots are ***zero***. This type of motion is said to have ***critical behavior***.

Case	Type of Roots	Type of Motion	Behavior
1	complex roots with negative real parts	asymptotically stable motion	significant
2	purely imaginary roots	constant or oscillatory motion	critical
3	complex roots with positive real parts	unstable motion	significant

The perturbed motion of the system about an equilibrium position is said to be ***infinitesimally stable*** if the roots of the characteristic equation have ***nonpositive real parts*** (cases 1 or 2). If the ***linearized*** equation exhibits ***significant behavior*** (cases 1 or 3) the ***stability characteristics*** of the ***linearized*** equation of motion are the ***same*** as for the ***complete nonlinear*** equation of motion. If the ***linearized*** equation exhibits ***critical behavior*** (case 2), then the stability of the nonlinear system ***cannot be determined*** from the ***linearized equation***.

It should be noted that, if the effects of ***friction*** and/or ***damping*** are included in the ***equation of motion***, the behavior identified by the ***linearized equation*** of motion is often ***changed*** from ***critical*** to ***significant***. However, if the inclusion of these effects still results in critical behavior, then the ***Liapunov direct method*** can be used. See reference 1 for details of this method.

Stability of Steady-State Equilibria for Multiple Degree-of-Freedom Systems

The dynamics of ***multiple degree-of-freedom*** rigid body systems are governed by a ***set of second-order***, ***nonlinear, ordinary differential equations of motion***. As presented above, the equations of motion can be

linearized about ***steady-state equilibrium positions***. Assuming there are ***no deviations*** of the ***external input*** (from their equilibrium values), the linearized equations can be written as follows.

$$\boxed{\{\dot{z}\} = \begin{Bmatrix} \{\Delta \dot{x}\} \\ \{\Delta \dot{y}\} \end{Bmatrix} = \left[\begin{array}{c|c} [\,0\,] & [\,I\,] \\ \hline [\,A_1\,] & [\,A_2\,] \end{array}\right] \begin{Bmatrix} \{\Delta x\} \\ \{\Delta y\} \end{Bmatrix} \triangleq [\,A\,]\{z\}} \tag{36}$$

Note that the elements in the coefficient matrix $[\,A\,]$ of Eq. (36) are constant.

The ***stability*** associated with ***infinitesimal motions*** of a ***multi-degree-of-freedom system*** about a ***steady-state equilibrium configuration*** can be determined by examining the ***eigenvalues*** of the matrix $[\,A\,]$. The eigenvalues are the ***roots*** of the ***characteristic equation*** of $[\,A\,]$. That is, they are the roots of the polynomial equation

$$\boxed{\det\left([\,A\,] - \lambda [\,I\,]\right) = 0} \tag{37}$$

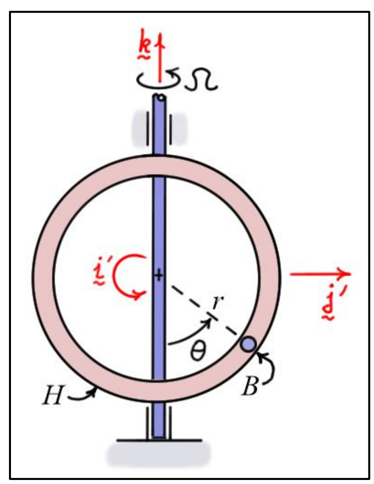
Eq. (37) represents an n^{th} order ***polynomial*** in λ if $[\,A\,]$ is an $n \times n$ matrix.

As with single degree-of-freedom systems, the types of ***infinitesimal motions*** of multi-degree-of-freedom systems about equilibrium configurations fall into ***three*** basic categories. The type of motion can be determined by examining the ***eigenvalues*** of $[\,A\,]$. The ***infinitesimal motions*** are ***asymptotically stable*** if all the eigenvalues have ***negative real parts*** and ***unstable*** if any of the eigenvalues have ***positive real parts***. These types of motion are said to have ***significant behavior***. The motion is ***constant*** or ***bounded oscillatory*** motion if the ***real parts*** of the eigenvalues are ***zero***. This type of motion is said to have ***critical behavior***.

If the ***linearized*** system of equations exhibits ***significant behavior***, the ***stability characteristics*** of the ***linearized*** system of equations of motion are the ***same*** as for the ***complete nonlinear*** equations of motion. If the ***linearized*** equations of motion exhibit ***critical behavior***, then the stability of the nonlinear system ***cannot be determined*** from the ***linearized equations***. The ***Liapunov direct method*** can be used for these systems. See reference 1 for details of this method.

<u>Example 4</u>:

The system of Example 1 is shown in the diagram. The linear equations that describe small motion of the ball relative to the hoop, H, about two of the three equilibrium positions of the ball for constant rotational rate Ω (as given by Eqs. (38) and (39)) are as follows.



Small motion about $\theta_e = 0$: $\boxed{\Delta\ddot{\theta} + (c/m)\Delta\dot{\theta} + \left(\frac{g}{r} - \Omega^2\right)\Delta\theta = 0}$

Small motion about $\theta_e = \cos^{-1}\left(\dfrac{g}{r\,\Omega^2}\right)$: $\boxed{\Delta\ddot{\theta} + (c/m)\Delta\dot{\theta} + \left(\dfrac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right)\Delta\theta = 0}$

Find:

a) Check the system stability at $\theta_e = 0$.

b) Check the system stability at $\theta_e = \cos^{-1}\left(g/r\,\Omega^2\right)$.

a) Applying ***Laplace transforms*** to the ***linear-approximate equation of motion*** gives the characteristic equation $\boxed{s^2 + \left(\frac{c}{m}\right)s + \left(\frac{g}{r} - \Omega^2\right) = 0}$. The roots of this equation are $\boxed{s_{1,2} = -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{g}{r} - \Omega^2\right)}}$. The roots of this equation fall into the following ***three cases***.

Case 1: $\boxed{\Omega^2 < \frac{g}{r}}$ and $\boxed{\left(\frac{c}{2m}\right)^2 > \left(\frac{g}{r} - \Omega^2\right)}$

$$\boxed{s_{1,2} = -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{g}{r} - \Omega^2\right)}}$$

Characteristic roots are ***real - valued*** and ***negative***. Solution to the approximate, linear equation of motion is ***over - damped*** and ***stable.***

Case 2: $\boxed{\Omega^2 < \frac{g}{r}}$ and $\boxed{\left(\frac{c}{2m}\right)^2 < \left(\frac{g}{r} - \Omega^2\right)}$

$$\boxed{s_{1,2} = -\left(\frac{c}{2m}\right) \pm j\sqrt{\left(\frac{g}{r} - \Omega^2\right) - \left(\frac{c}{2m}\right)^2}}$$

Characteristic roots are ***complex - valued*** with ***negative real parts***. Solution to the approximate, linear equation of motion is ***under - damped*** and ***stable.***

Case 3: $\boxed{\Omega^2 > \frac{g}{r}}$

$$\boxed{s_{1,2} = -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \left(\Omega^2 - \frac{g}{r}\right)}}$$

Characteristic roots are ***real - valued*** with ***one positive*** and ***one negative.*** Solution to the approximate, linear equation of motion is ***unstable.***

In all three cases, the motion described by the ***linear-approximate equation*** represents ***significant behavior***, so the ***stability conclusions*** also apply to the ***nonlinear system***.

b) Applying ***Laplace transforms*** to the ***linear-approximate equation of motion*** gives the characteristic equation $\boxed{s^2 + (c/m)s + \left(\frac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right) = 0}$. The roots of this equation are $\boxed{s_{1,2} = -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right)}}$. As noted above, this equilibrium position ***exists*** and is different from the first position (i.e. $\theta = 0$) ***only if*** $\Omega^2 > \frac{g}{r}$. As a result, the types of roots this equation represents fall into the following two cases.

Case 1: $\boxed{\Omega^2 > \frac{g}{r}}$ and $\boxed{\left(\frac{c}{2m}\right)^2 > \left(\frac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right)}$

$$s_{1,2} = -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right)}$$

Characteristic roots are ***real - valued*** and ***negative***. Solution to the approximate, linear equation of motion is ***over - damped*** and ***stable.***

Case 2: $\Omega^2 > \frac{g}{r}$ and $\left(\frac{c}{2m}\right)^2 < \left(\frac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right)$

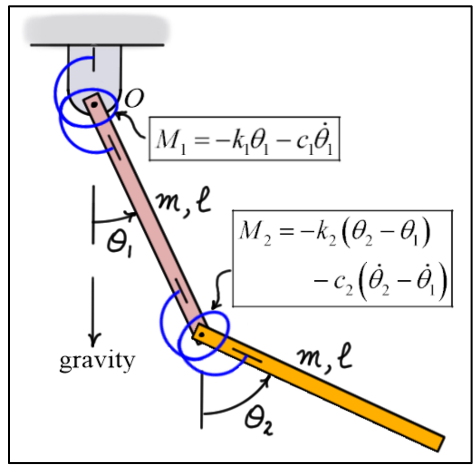
$$s_{1,2} = -\left(\frac{c}{2m}\right) \pm j\sqrt{\left(\frac{r^2\,\Omega^4 - g^2}{r^2\,\Omega^2}\right) - \left(\frac{c}{2m}\right)^2}$$

Characteristic roots are ***complex - valued*** with ***negative real parts***. Solution to the approximate, linear equation of motion is ***under - damped*** and ***stable.***

So, if this equilibrium position ***exists***, it is ***stable***, and it ***exists*** only when the equilibrium position $\theta_e = 0$ is ***unstable***. Also, the motions described by the ***linear-approximate equation*** in both cases represent ***significant behavior***, so the ***stability conclusions*** also apply to the ***nonlinear system***.

Example 5:

The diagram shows the double pendulum system of Examples 2 and 3. Example 3 showed the linear approximate equations of motion of the system about $(\theta_1, \theta_2)_{eq} = (0,0)$ can be written the following matrix form.



$$\{\dot{z}\} = \begin{Bmatrix} \begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix} \\ \begin{Bmatrix} \Delta\ddot{\theta}_1 \\ \Delta\ddot{\theta}_2 \end{Bmatrix} \end{Bmatrix} = [A] \begin{Bmatrix} \begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \end{Bmatrix} \\ \begin{Bmatrix} \Delta\dot{\theta}_1 \\ \Delta\dot{\theta}_2 \end{Bmatrix} \end{Bmatrix} = [A]\{z\}$$

with

$$[A] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -\left(\frac{18g}{7\ell} + \frac{12k_1}{7m\ell^2} + \frac{30k_2}{7m\ell^2}\right) & \left(\frac{9g}{7\ell} + \frac{30k_2}{7m\ell^2}\right) & -\left(\frac{12c_1}{7m\ell^2} + \frac{30c_2}{7m\ell^2}\right) & \frac{30c_2}{7m\ell^2} \\ \left(\frac{27g}{7\ell} + \frac{18k_1}{7m\ell^2} + \frac{66k_2}{7m\ell^2}\right) & -\left(\frac{24g}{7\ell} + \frac{66k_2}{7m\ell^2}\right) & \frac{18c_1}{7m\ell^2} + \frac{66c_2}{7m\ell^2} & -\frac{66c_2}{7m\ell^2} \end{array}\right]$$

Given:

$mg = 5$ (lb), $g = 32.2$ (ft/s^2) , $\ell = 2$ (ft)

$k_1 = 1000$ (ft-lb/rad), $k_2 = 300$ (ft-lb/rad), $c_1 = 5$ (ft-lb-s/rad), $c_2 = 0.5$ (ft-lb-s/rad)

Find:

a) Find $\lambda_i \ (i = 1,2,3,4)$ the roots of the characteristic equation.

b) Determine if the motion about $\left(\theta_1,\theta_2\right)_{eq}=\left(0,0\right)$ is stable.

Solution:

Using the given values, matrix $\left[\,A\,\right]$ is

$$\left[\,A\,\right]=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4871.4 & 2090.7 & -17.25 & 3.45 \\ 8756.1 & -4609.2 & 28.29 & -7.59 \end{bmatrix}$$

The characteristic equation associated with $\left[\,A\,\right]$ is

$$\det\left(\left[\,A\,\right]-\lambda\left[\,I\,\right]\right)=0=\det\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -4871.4 & 2090.7 & -\left(17.25+\lambda\right) & 3.45 \\ 8756.1 & -4609.2 & 28.29 & -\left(7.59+\lambda\right) \end{bmatrix}$$

$$=-\lambda\det\begin{bmatrix} -\lambda & 0 & 1 \\ 2090.7 & -\left(17.25+\lambda\right) & 3.45 \\ -4609.2 & 28.29 & -\left(7.59+\lambda\right) \end{bmatrix}+\det\begin{bmatrix} 0 & -\lambda & 1 \\ -4871.4 & 2090.7 & 3.45 \\ 8756.1 & -4609.2 & -\left(7.59+\lambda\right) \end{bmatrix}$$

$$=\lambda^2\det\begin{bmatrix} -\left(17.25+\lambda\right) & 3.45 \\ 28.29 & -\left(7.59+\lambda\right) \end{bmatrix}-\lambda\det\begin{bmatrix} 2090.7 & -\left(17.25+\lambda\right) \\ -4609.2 & 28.29 \end{bmatrix}$$

$$+\lambda\det\begin{bmatrix} -4871.4 & 3.45 \\ 8756.1 & -\left(7.59+\lambda\right) \end{bmatrix}+\det\begin{bmatrix} -4871.4 & 2090.7 \\ 8756.1 & -4609.2 \end{bmatrix}$$

$$=\lambda^2\left[33.327+24.84\lambda+\lambda^2\right]+\lambda\left[20362.8+4609.2\lambda\right]$$

$$+\lambda\left[6765.43+4871.4\lambda\right]+\left(4.14688\times10^6\right)$$

$$\Rightarrow \boxed{\lambda^4+\left(24.84\right)\lambda^3+\left(9513.93\right)\lambda^2+\left(27128.2\right)\lambda+4.14688\times10^6=0}$$

The roots of this equation are

$$\boxed{\lambda_{1,2}=-11.5022\pm j\,94.2296} \qquad \boxed{\lambda_{3,4}=-0.917835\pm j\,21.432}$$

b) The roots of the characteristic equation ***all*** have ***negative real parts***, so the linear-approximate system and the original nonlinear system are both ***stable***.

Natural Frequencies and Mode Shapes of Undamped Systems

The linear-approximate equations of motion about an equilibrium position can also be used to calculate the ***natural frequencies*** and ***mode shapes*** for ***multiple degree-of-freedom***, rigid-body systems. These frequencies and mode shapes describe ***undamped***, ***small motions*** of the system about the steady-state equilibrium positions. These small motions can be written as a linear combination of contributions from each of the mode shapes. The so-called modal coordinates determine how much each mode shape is contributing to the motion at any time. (For more details on modal coordinates, see reference 4.) The following paragraphs describe how the ***natural frequencies*** and ***mode shapes*** can be calculated from the ***linearized equations of motion***. It is assumed that the system's motion is ***infinitesimally stable*** about the equilibrium position.

Recall from above that the linearized equations of motion about an equilibrium position can be written either as a set of ***second-order equations*** or a set of ***first-order*** equations as indicated in Eqs. (40) and (41) below.

$$\left[\,M\,\right]\{\Delta\ddot{x}\} + \left[\,C\,\right]\{\Delta\dot{x}\} + \left[\,K\,\right]\{\Delta x\} = \{\Delta u\} \tag{40}$$

$$\begin{Bmatrix}\{\Delta\dot{x}\}\\ \{\Delta\dot{y}\}\end{Bmatrix} = \left[\begin{array}{c|c}[\,0\,] & [\,I\,]\\ \hline [\,A_1\,] & [\,A_2\,]\end{array}\right]\begin{Bmatrix}\{\Delta x\}\\ \{\Delta y\}\end{Bmatrix} + \left[\begin{array}{c}[\,0\,]\\ \hline [\,B\,]\end{array}\right]\{\Delta u\} \tag{41}$$

Recall in the above equations that, $\left[\,M\,\right]$, $\left[\,C\,\right]$, and $\left[\,K\,\right]$ represent the system's ***mass***, ***damping***, and ***stiffness matrices***, vector $\{\Delta x\}$ represents ***changes*** in all the ***generalized coordinates*** from their equilibrium values, vector $\{\Delta y\} \triangleq \{\Delta\dot{x}\}$, and vector $\{\Delta u\}$ represents ***changes*** in the ***external excitations*** from their equilibrium values.

Using Eq. (40) and setting the ***damping matrix*** and the ***changes*** in the ***external excitations*** to ***zero*** gives an equation of motion representing ***free***, ***undamped response*** of the system near the equilibrium.

$$\left[\,M\,\right]\{\Delta\ddot{x}\} + \left[\,K\,\right]\{\Delta x\} = \{0\} \qquad (\textbf{\textit{free, undamped response}})$$

Following the pattern for single degree-of-freedom systems, a solution is sought of the form

$$\{\Delta x\} = e^{j\omega t}\{v\}$$

This equation describes a ***steady-state***, ***undamped*** solution, with $\{v\}$ representing the ***mode shape*** of the oscillations. Substituting this into the differential EOM gives

$$\left(\left[\,K\,\right] - \omega^2\left[\,M\,\right]\right)\{v\} = \{0\} \tag{42}$$

The problem now is to find ω^2 and $\{v\}$ that satisfy this ***algebraic*** equation. This is called an ***eigenvalue problem*** – the values of $\omega^2 \triangleq \lambda$ are the squares of the ***eigenvalues***, and the associated vectors $\{v\}$ are the ***eigenvectors***.

For Eq. (42) to have a ***non-zero*** solution for $\{v\}$, the ***determinant*** of the ***coefficient matrix*** must be ***zero***.

$$\det\left(\left[\,K\,\right] - \omega^2\left[\,M\,\right]\right) = \{0\}$$

For $N \times N$ matrices $\left[\,M\,\right]$ and $\left[\,K\,\right]$, the solution to this equation yields N values for ω^2, and for each value of ω^2 there is an associated eigenvector $\{v\}$. These are the N ***undamped***, ***natural frequencies*** of the system and their associated ***mode shapes***. The mode shapes describe the "shape" of the motion that is associated with that frequency.

Note that setting the determinant of the coefficient matrix in Eq. (42) to zero means that a ***unique solution*** for any eigenvector $\{v\}$ is ***not possible***. The ***relative values*** of the elements of $\{v\}$ can be found, but the overall magnitude of the vector is not. Choosing the value of one of the elements (a value of one, for example) allows the other elements of $\{v\}$ to be found.

Switching now to Eq. (41), it is helpful to make the following observations. The equations can be split into two sets. The first set defines the vector $\{\Delta y\} \triangleq \{\Delta \dot{x}\}$. The second set is the same as those in Eq. (40) after solving for $\{\Delta \ddot{x}\}$. Consequently, Eq. (41) can be rewritten as follows.

$$\boxed{\begin{Bmatrix} \{\Delta \dot{x}\} \\ \{\Delta \dot{y}\} \end{Bmatrix} = \left[\begin{array}{c|c} [0] & [I] \\ \hline [A_1] & [A_2] \end{array}\right] \begin{Bmatrix} \{\Delta x\} \\ \{\Delta y\} \end{Bmatrix} + \begin{bmatrix} [0] \\ [B] \end{bmatrix} \{\Delta u\} = \left[\begin{array}{c|c} [0] & [I] \\ \hline [M]^{-1}[K] & [M]^{-1}[C] \end{array}\right] \begin{Bmatrix} \{\Delta x\} \\ \{\Delta y\} \end{Bmatrix} + \begin{bmatrix} [0] \\ [M]^{-1} \end{bmatrix} \{\Delta u\}}$$

Setting the ***damping matrix*** and the ***changes*** in the ***external excitation*** to ***zero*** gives

$$\boxed{\begin{Bmatrix} \{\Delta \dot{x}\} \\ \{\Delta \dot{y}\} \end{Bmatrix} = \left[\begin{array}{c|c} [0] & [I] \\ \hline [M]^{-1}[K] & [0] \end{array}\right] \begin{Bmatrix} \{\Delta x\} \\ \{\Delta y\} \end{Bmatrix} \triangleq [A] \begin{Bmatrix} \{\Delta x\} \\ \{\Delta y\} \end{Bmatrix}} \qquad (43)$$

The undamped natural frequencies and mode shapes can now be found from Eq. (43) by calculating the ***eigenvalues*** and ***eigenvectors*** of coefficient matrix $[A]$. With the equations in this form, the eigenvalues are the natural frequencies, not the squares of the natural frequencies. Also, with only masses and stiffnesses present in the equations, the system response is oscillatory making the eigenvalues purely imaginary.

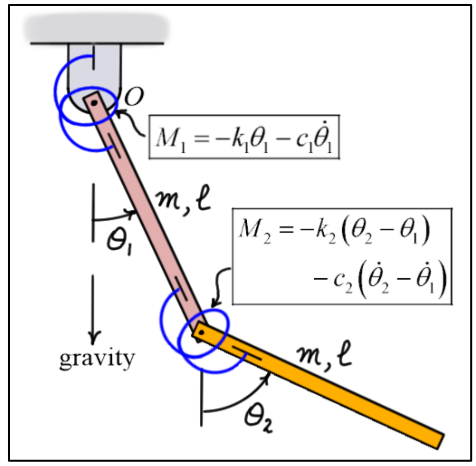
The eigenvector associated with an eigenvalue λ_i takes the form

$$\boxed{\{v_i\} = \begin{Bmatrix} \{\hat{v}_i\} \\ \hline \lambda_i \{\hat{v}_i\} \end{Bmatrix}}$$

The eigenvector $\{v_i\}$ can be ***partitioned*** into two halves. The ***upper half*** $\{\hat{v}_i\}$ represents the mode shape, and the ***lower half*** is the product of the eigenvalue and the mode shape. The eigenvector is ***real-valued*** for real eigenvalues and ***complex-valued*** for complex eigenvalues. Also, eigenvectors that correspond to complex conjugate eigenvalues are themselves complex conjugates of each other. When mass and stiffness are present in the equations of motion but no damping, the eigenvalues and mode shapes are purely imaginary. See example 7 below.

Example 6:

The diagram shows the double pendulum system of Examples 2 and 3. Example 2 showed the linear approximate equations of motion of the system about $(\theta_1,\theta_2)_{eq}=(0,0)$ can be written the following matrix form.



$$[M]\begin{Bmatrix}\Delta\ddot{\theta}_1\\ \Delta\ddot{\theta}_2\end{Bmatrix}+[C]\begin{Bmatrix}\Delta\dot{\theta}_1\\ \Delta\dot{\theta}_2\end{Bmatrix}+[K]\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

with

$$[M]\triangleq\begin{bmatrix}\frac{4}{3}m\ell^2 & \frac{1}{2}m\ell^2\\ \frac{1}{2}m\ell^2 & \frac{1}{3}m\ell^2\end{bmatrix}\qquad [C]\triangleq\begin{bmatrix}c_1+c_2 & -c_2\\ -c_2 & c_2\end{bmatrix}$$

$$[K]\triangleq\begin{bmatrix}k_1+k_2+\frac{3}{2}mg\ell & -k_2\\ -k_2 & k_2+\frac{1}{2}mg\ell\end{bmatrix}$$

Given:

$mg=5$ (lb), $g=32.2$ (ft/s^2), $\ell=2$ (ft), $k_1=1000$ (ft-lb/rad), and $k_2=300$ (ft-lb/rad)

Find:

a) Find $\omega_i\ (i=1,2)$ the natural frequencies of the system about the equilibrium position $\theta_1=\theta_2=0$.
b) Find the mode shapes associated with these frequencies.
c) Describe the motion associated with each mode shape.

Solution:

a) Using the values given above, the mass and stiffness matrices are

$$[M]=\begin{bmatrix}0.828157 & 0.310559\\ 0.310559 & 0.207039\end{bmatrix}\qquad [K]=\begin{bmatrix}1315 & -300\\ -300 & 305\end{bmatrix}$$

The eigenvalues for the system are found by setting

$$\det([K]-\lambda[M])=\begin{bmatrix}(1315-0.828157\lambda) & -(300+0.310559\lambda)\\ -(300+0.310559\lambda) & (305-0.207039\lambda)\end{bmatrix}=0$$

$$\Rightarrow (0.075014)\lambda^2-(711.18)\lambda+311075=0$$

The roots of this equation are the squares of the natural frequencies.

Roots: $\lambda_{1,2}=\begin{cases}459.697\\ 9020.93\end{cases}$

Natural Frequencies: $\omega_{1,2}=\sqrt{\lambda_{1,2}}=\begin{cases}21.4405\ \text{(r/s)}\\ 94.9786\ \text{(r/s)}\end{cases}$ $\quad f_{1,2}=\dfrac{\omega_{1,2}}{2\pi}=\begin{cases}3.4124\ \text{(Hz)}\\ 15.1163\ \text{(Hz)}\end{cases}$

b) ***Mode 1:*** $\omega_1\approx 21.4$ (r/s), $f_1\approx 3.41$ (Hz)

To find the mode shape associated with this frequency, set

$$\left([K]-\lambda_1[M]\right)\{v_1\}=\begin{bmatrix}934.299 & -442.763\\ -442.763 & 209.825\end{bmatrix}\begin{Bmatrix}v_{11}\\ v_{12}\end{Bmatrix}=\{0\}$$

Setting $v_{11}=1$ and solving either equation for v_{12} gives $\boxed{v_{12}=2.110}$

So, the mode shape associated with this frequency is $\boxed{\{v_1\}=\begin{Bmatrix}1\\ 2.110\end{Bmatrix}}$

Mode 2: $\boxed{\omega_2\approx 95.0\text{ (r/s)}}$, $\boxed{f_2\approx 15.1\text{ (Hz)}}$

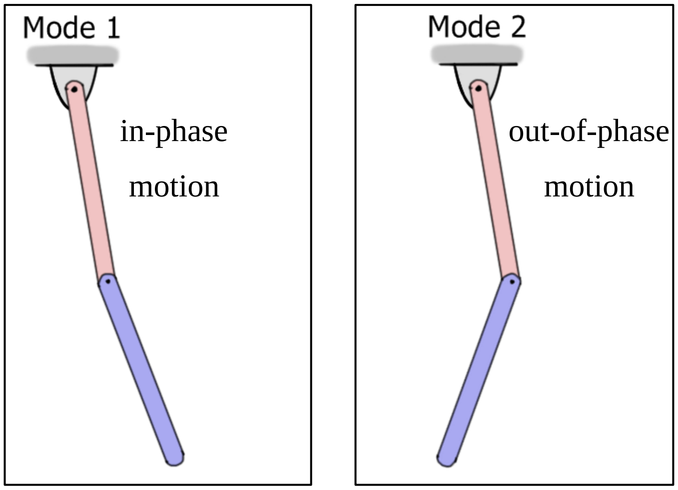
To find the mode shape associated with this frequency, set

$$\left([K]-\lambda_2[M]\right)\{v_2\}=\begin{bmatrix}-6155.75 & -3101.53\\ -3101.53 & -1562.68\end{bmatrix}\begin{Bmatrix}v_{21}\\ v_{22}\end{Bmatrix}=\{0\}$$

Setting $v_{21}=1$ and solving either equation for v_{22} gives $\boxed{v_{22}=-1.985}$

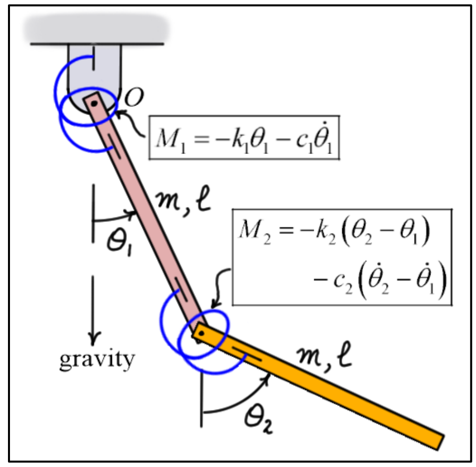
So, the mode shape associated with this frequency is $\boxed{\{v_2\}=\begin{Bmatrix}1\\ -1.985\end{Bmatrix}}$

c) In mode 1, as the first link moves, for example, through an angle of 1 degree, the second link rotates in the ***same direction*** through an angle of 2.11 degrees. The two links move ***in phase*** and reach their ***peak amplitudes*** at the ***same time***. In mode 2, as the first link moves, for example, through an angle of 1 degree, the second link rotates in the ***opposite direction*** through an angle of 1.985 degrees. The motions of the two links are ***180 degrees out-of-phase*** and reach their peak amplitudes (opposite in sign) at the same time.



Example 7:

The diagram shows the double pendulum system of Examples 2 and 3. Example 3 showed the linear approximate equations of motion of the system about $(\theta_1,\theta_2)_{eq}=(0,0)$ can be written the following matrix form.



$$\{\dot{z}\}=\begin{Bmatrix}\begin{Bmatrix}\Delta\dot{\theta}_1\\ \Delta\dot{\theta}_2\end{Bmatrix}\\ \begin{Bmatrix}\Delta\ddot{\theta}_1\\ \Delta\ddot{\theta}_2\end{Bmatrix}\end{Bmatrix}=[A]\begin{Bmatrix}\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\end{Bmatrix}\\ \begin{Bmatrix}\Delta\dot{\theta}_1\\ \Delta\dot{\theta}_2\end{Bmatrix}\end{Bmatrix}=[A]\{z\}$$

with

$$[A]=\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline -\left(\frac{18g}{7\ell}+\frac{12k_1}{7m\ell^2}+\frac{30k_2}{7m\ell^2}\right) & \left(\frac{9g}{7\ell}+\frac{30k_2}{7m\ell^2}\right) & -\left(\frac{12c_1}{7m\ell^2}+\frac{30c_2}{7m\ell^2}\right) & \frac{30c_2}{7m\ell^2}\\ \left(\frac{27g}{7\ell}+\frac{18k_1}{7m\ell^2}+\frac{66k_2}{7m\ell^2}\right) & -\left(\frac{24g}{7\ell}+\frac{66k_2}{7m\ell^2}\right) & \frac{18c_1}{7m\ell^2}+\frac{66c_2}{7m\ell^2} & -\frac{66c_2}{7m\ell^2}\end{array}\right]$$

Given:

$mg=5$ (lb), $g=32.2$ (ft/s^2), $\ell=2$ (ft), $k_1=1000$ (ft-lb/rad), and $k_2=300$ (ft-lb/rad)

Find:

a) Find $\omega_i\ (i=1,2)$ the natural frequencies of the system about the equilibrium position $\theta_1=\theta_2=0$.

b) Find the mode shapes associated with these frequencies.

Solution:

a) Substituting the given values into the coefficient matrix with $c_1=c_2=0$ gives

$$[A]=\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -4871.4 & 2090.7 & 0 & 0\\ 8756.1 & -4609.2 & 0 & 0\end{bmatrix}$$

Using MATLAB®, the eigenvalues of $[A]$ are

$\lambda_{1,2}=\pm j\,94.9784365$ $\quad\lambda_{3,4}=\pm j\,21.4405363$

Natural Frequencies: $\omega_{1,2}=\begin{cases}21.4405\text{ (r/s)}\\ 94.9784\text{ (r/s)}\end{cases}$ $\quad f_{1,2}=\dfrac{\omega_{1,2}}{2\pi}=\begin{cases}3.4124\text{ (Hz)}\\ 15.1163\text{ (Hz)}\end{cases}$

The eigenvectors associated with these eigenvalues are (as calculated using MATLAB®)

$$\{v_1\} = [\,j0.00473720628, -j0.00940213975, -0.449932446, 0.893000533]^T$$

$$\{v_2\} = [-j0.00473720628, j0.00940213975, -0.449932446, 0.893000533]^T$$

$$\{v_3\} = [-j0.0199519042, -j0.0421016328, 0.427779528, 0.902681588]^T$$

$$\{v_4\} = [\,j0.0199519042, j0.0421016328, 0.427779528, 0.902681588]^T$$

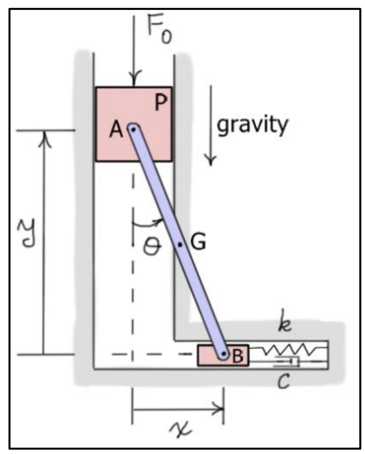
Interpretation of the mode shapes can be done by examining the first two or the last two elements of the eigenvectors. The last two elements of eigenvectors 3 and 4 describe a mode shape where the second entry is $(0.902681588/0.427779528) \approx 2.110$ times the first, and the last two elements of eigenvectors 1 and 2 describe a mode shape where the second entry is $(0.893000533/-0.449932446) \approx -1.985$ times the first. So, as in the previous example,

Mode 1: $\boxed{\omega_1 \approx 21.4 \text{ (r/s)}}$, $\boxed{f_1 \approx 3.41 \text{ (Hz)}}$, and $\boxed{\{\hat{v}_1\} = \begin{Bmatrix} 1 \\ 2.110 \end{Bmatrix}}$ (mode shape)

Mode 2: $\boxed{\omega_2 \approx 95.0 \text{ (r/s)}}$, $\boxed{f_2 \approx 15.1 \text{ (Hz)}}$. and $\boxed{\{\hat{v}_2\} = \begin{Bmatrix} 1 \\ -1.985 \end{Bmatrix}}$ (mode shape)

Exercises:

8.1 The system shown consists of slender bar *AB* of mass *m* and length ℓ and a piston *P* of mass m_p. The system is loaded by the force F_0 and gravity. A spring and damper are attached to the light slider at *B*. The spring is unstretched when $x = 0$. Neglecting friction and using the angle θ as the generalized coordinate, it can be shown that the equation of motion can be written as follows.



$$\boxed{\begin{aligned}&\left[\tfrac{1}{3}m\ell^2 + m_p\ell^2 S_\theta^2\right]\ddot{\theta} + \left[m_p\ell^2 S_\theta C_\theta\right]\dot{\theta}^2 + \left[c\ell^2 C_\theta^2\right]\dot{\theta} + k\ell^2 S_\theta C_\theta \\ &\qquad -\left[m_p g + \tfrac{1}{2}mg + F_0\right]\ell S_\theta = 0\end{aligned}}$$

a) Show that $\theta = 0$ is the only equilibrium position of the bar for the following data.
   $mg = 4$ (lb); $m_P g = 3$ (lb); $\ell = 2$ (ft); $k = 15$ lb/ft; $c = 0.2$; $F_0 = 40$ lb

b) Find the linear approximate equation of motion about $\theta = 0$.

c) Determine the stability of small motion about $\theta = 0$.

d) Find the range of spring stiffness values for which small motion about $\theta = 0$ is stable.

e) Show that the system is stable for some stiffness in that range.

f) Are there other equilibrium positions for the stiffness chosen in part (e)?

g) If there is a new equilibrium position, determine the stability of small motion about it.

Answers:

b) $\left[\frac{1}{3}m\ell^2\right]\Delta\ddot{\theta}+\left[c\,\ell^2\right]\Delta\dot{\theta}+\left[k\ell-\left[m_p g+\frac{1}{2}mg+F_0\right]\right]\ell\Delta\theta=0$

c) Characteristic Eq.: $0.165631s^2+0.8s-30=0$ Roots: $s_{1,2}=11.258,-16.088$

The behavior is ***significant*** and ***unstable***, so the linear and nonlinear equations are ***unstable***.

d) $k>22.5$ (ft-lb/rad)

e) $k=25$ (ft-lb/rad) Characteristic Eq.: $0.165631s^2+0.8s+10=0$ Roots: $s_{1,2}=-2.415\pm j7.385$

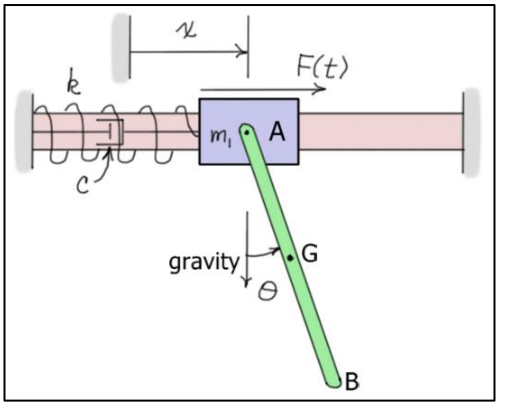
The behavior is ***significant*** and ***stable***, so the linear and nonlinear equations are ***stable***.

f) Yes, $\theta\approx 25.8419$ (deg)

g) Characteristic Eq: $0.236439s^2+0.648s-19=0$ Roots: $s_{1,2}=7.6981,-10.439$

The behavior is ***significant*** and ***unstable***, so the linear and nonlinear equations are ***unstable***.

8.2 The system shown consists of a mass m_1 that translates along a fixed horizontal bar and a uniform slender bar *AB* that is pinned to m_1 at *A*. Bar *AB* has mass m_2 and length ℓ. Mass m_1 is attached to the fixed support by a spring of stiffness *k* and a linear viscous damper with coefficient *c*. The spring is unstretched when $x=0$. The system is driven by gravity and the force $F(t)=F_0\sin(\omega t)$ applied to m_1. Neglecting friction and using the variables *x* and θ as the generalized coordinates, it can be shown that the equations of motion can be written as follows.



$$\left(m_1+m_2\right)\ddot{x}+\left(\tfrac{1}{2}m_2\ell C_\theta\right)\ddot{\theta}-\left(\tfrac{1}{2}m_2\ell S_\theta\right)\dot{\theta}^2+c\dot{x}+kx=F(t)$$
$$\left(\tfrac{1}{3}m_2\ell^2\right)\ddot{\theta}+\left(\tfrac{1}{2}m_2\ell C_\theta\right)\ddot{x}+\tfrac{1}{2}m_2 g\ell S_\theta=0$$

a) Show that $x=\theta=0$ is an equilibrium position for the system.

b) Find the linear approximate equations of motion for small motion about this position.

c) Given that $m_1 g=10$ (lb), $m_2 g=5$ (lb), $\ell=2$ (ft), and $k=300$ (lb/ft), find the natural frequencies and mode shapes associated with linearized model. Describe the motion of each mode.

Answers:

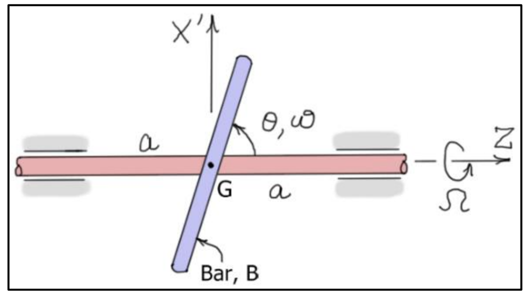
b) $$\begin{bmatrix} m_1+m_2 & \frac{1}{2}m_2\ell \\ \frac{1}{2}m_2\ell & \frac{1}{3}m_2\ell^2 \end{bmatrix}\begin{Bmatrix}\Delta\ddot{x}\\ \Delta\ddot{\theta}\end{Bmatrix}+\begin{bmatrix} c & 0 \\ 0 & 0\end{bmatrix}\begin{Bmatrix}\Delta\dot{x}\\ \Delta\dot{\theta}\end{Bmatrix}+\begin{bmatrix} k & 0 \\ 0 & \frac{1}{2}m_2 g\ell\end{bmatrix}\begin{Bmatrix}\Delta x\\ \Delta\theta\end{Bmatrix}=\begin{Bmatrix}F(t)\\ 0\end{Bmatrix}$$

c) $\omega_{1,2}\approx\begin{cases}4.89\text{ (r/s)}\\ 29.44\text{ (r/s)}\end{cases}$ $f_{1,2}=\frac{\omega_{1,2}}{2\pi}\approx\begin{cases}0.7784\text{ (Hz)}\\ 4.686\text{ (Hz)}\end{cases}$ $\left\{\hat{v}_1\right\}=\begin{Bmatrix}1\\ 77.8\end{Bmatrix}$ $\left\{\hat{v}_2\right\}=\begin{Bmatrix}1\\ -0.7715\end{Bmatrix}$

Mode 1: m_1 and m_2 move in-phase – mass moves right as bar rotates counterclockwise, and vice versa. Very small mass motion relative to the bar's rotation. Mode 2: m_1 and m_2 move out-of-phase

– mass moves left as bar rotates counterclockwise, and vice versa. Mass motion and the bar's rotation have similar magnitude.

8.3 The system shown consists of a uniform slender bar B of length ℓ and mass m that is pinned at its mass-center through the center of a shaft of mass m_s and radius r. The rotation of the shaft about the Z-axis is described by the angle ϕ $(\dot{\phi}=\Omega)$, and the rotation of the bar B about the Y' axis is described by the angle θ $(\dot{\theta}=\omega)$. A motor torque M_ϕ is applied to the shaft about the Z-axis, and a motor torque M_θ is applied to B about the Y' axis.



Using the angles θ and ϕ as the generalized coordinates, it can be shown that the equations of motion of the system can be written as follows.

$$\left[\tfrac{1}{2}m_s r^2+\tfrac{1}{12}m\ell^2 S_\theta^2\right]\ddot{\phi}+\left(\tfrac{1}{6}m\ell^2 S_\theta C_\theta\right)\dot{\theta}\dot{\phi}=M_\phi$$
$$\left(\tfrac{1}{12}m\ell^2\right)\ddot{\theta}-\left(\tfrac{1}{12}m\ell^2 S_\theta C_\theta\right)\dot{\phi}^2=M_\theta$$

a) Given $M_\theta=0$ and $\dot{\phi}=\Omega=\text{constant}$, find the equilibrium positions for the angle θ.

b) Find the linear approximate equations of motion for θ about these positions.

c) Determine the stability of small motions about each of these positions.

Answers:

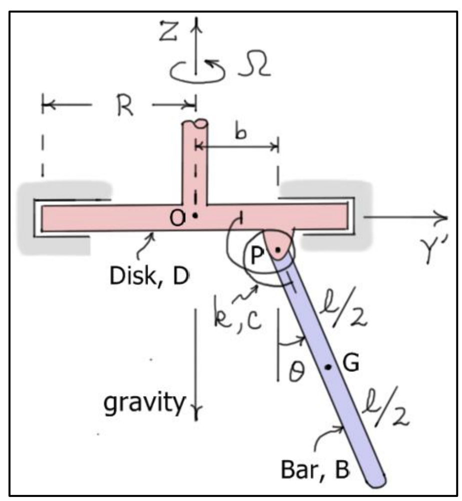
a) $\theta_{eq}=0$; $\theta_{eq}=90$ (deg)

b) $\theta_{eq}=0 \;\Rightarrow\; \boxed{\Delta\ddot{\theta}-\left(\Omega^2\right)\Delta\theta=0}$ and $\theta_{eq}=90$ (deg) $\;\Rightarrow\; \boxed{\Delta\ddot{\theta}+\left(\Omega^2\right)\Delta\theta=0}$

c) $\theta_{eq}=0$ is an unstable equilibrium, and $\theta_{eq}=90$ (deg) is a stable equilibrium

The behavior is ***critical***, so the system is only ***infinitesimally stable***.

8.4 The system shown consists of a disk D of mass m_d and radius R, and a uniform slender bar B of mass m and length ℓ. The rotation of the disk about the Z axis is described by the angle ϕ $(\dot{\phi}=\Omega)$, and the rotation of the bar B about the X' axis is described by the angle θ $(\dot{\theta}=\omega)$. A linear rotational spring-damper is located between B and D at the pin P. The spring has stiffness k and is unstretched when $\theta=0$. The damper has coefficient c. A motor torque M_ϕ is applied to the disk about the Z axis, and a motor torque M_θ is applied to B about the X' axis.



Using angles θ and ϕ as generalized coordinates, it can be shown that the equations of motion can be written as follows.

$$\left[\tfrac{1}{2}m_d R^2 + m\left(b+\tfrac{1}{2}\ell S_\theta\right)^2 + \tfrac{1}{12}m\ell^2 S_\theta^2\right]\ddot{\phi} + (mb\ell C_\theta + \tfrac{2}{3}m\ell^2 S_\theta C_\theta)\dot{\theta}\dot{\phi} = M_\phi$$

$$\left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta} - \left[\tfrac{1}{2}mb\ell C_\theta + \tfrac{1}{3}m\ell^2 S_\theta C_\theta\right]\dot{\phi}^2 + c\dot{\theta} + k\theta + \tfrac{1}{2}mg\ell S_\theta = M_\theta$$

The following data is given.

$M_\theta = 0$, $\dot{\phi} = \Omega = \text{constant} = 4\pi$ (rad/s), $b = 0$, $mg = 5$ (lb), $\ell = 2$ (ft)

$k = 10$ (ft-lb/rad), $c = 0.5$ (ft-lb-s/rad)

a) Show that the equilibrium position for the angle θ is approximately $\theta_{eq} \approx 58.7$ (deg).

b) Find the linear approximate equation of motion for the angle θ about this position.

c) Determine the stability of small motions about the equilibrium.

Answers:

a) $\theta \approx 58.7$ (deg) is a solution to the equilibrium equation: $-32.694 S_\theta C_\theta + 10\theta + 5S_\theta = 0$

b) $\Delta\ddot{\theta} + (2.415)\Delta\dot{\theta} + (133.52)\Delta\theta = 0$

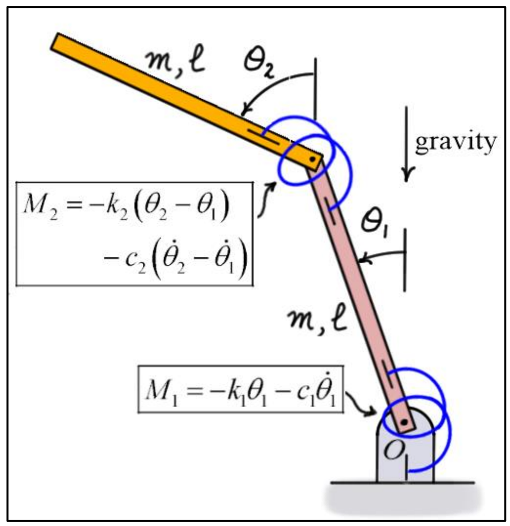
c) Characteristic Eq.: $s^2 + (2.415)s + (133.52) = 0$ Roots: $s_{1,2} \approx -1.2075 \pm j11.492$

The behavior is ***significant*** and ***stable***, so the linear and nonlinear equations are ***stable***.

8.5 The system of Example #5 is inverted as shown in the diagram. The equations of motion for this system are identical to those of Example #5, except all terms associated with gravity have opposite sign. Complete the following, given that

$mg = 5$ (lb), $g = 32.2$ (ft/s^2), $\ell = 2$ (ft)

$c_1 = 5$ (ft-lb-s/rad), $c_2 = 0.5$ (ft-lb-s/rad)



a) Determine the stability of motion around the equilibrium position $(\theta_1, \theta_2)_{eq} = (0,0)$ given

$k_1 = 1000$ (ft-lb/rad), $k_2 = 300$ (ft-lb/rad)

b) Determine the stability of motion around the equilibrium position $(\theta_1, \theta_2)_{eq} = (0,0)$ given

$k_1 = 100$ (ft-lb/rad), $k_2 = 5$ (ft-lb/rad)

c) Find the natural frequencies and mode shapes associated with the linearized equation of motion of part (a). Describe the motion each mode.

Answers:

a) Eigenvalues: $\lambda_{1,2} = -11.506 \pm j93.330$ $\lambda_{3,4} = -0.914116 \pm j20.855$

The behavior is ***significant*** and ***stable***, so the linear and nonlinear systems are ***stable***.

b) Eigenvalues: $\lambda_{1,2} = -11.413 \pm j12.170$, $\lambda_3 = -2.4939$, $\lambda_4 = 0.48008$

The behavior is ***significant*** and ***unstable***, so the linear and nonlinear systems are ***unstable***.

c) $\omega_{1,2} \approx \begin{cases} 20.87 \text{ (r/s)} \\ 94.09 \text{ (r/s)} \end{cases}$ $f_{1,2} = \dfrac{\omega_{1,2}}{2\pi} \approx \begin{cases} 3.32 \text{ (Hz)} \\ 14.97 \text{ (Hz)} \end{cases}$ $\{\hat{v}_1\} = \begin{Bmatrix} 1 \\ 2.124 \end{Bmatrix}$ $\{\hat{v}_2\} = \begin{Bmatrix} 1 \\ -1.982 \end{Bmatrix}$

Mode 1: Links move in-phase – as the first link rotates counterclockwise, the second also rotates counterclockwise, and vice versa. Second link rotates just over twice as fast. Mode 2: Links move out-of-phase – as the first link rotates counterclockwise, the second also rotates clockwise, and vice versa. Second link rotates just under twice as fast.

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