Introductory Control Systems
Exercises #2 – Mathematical Models and Linearization

Mathematical Models

Mathematical models of system dynamics are often developed from ***physical principles*** that are expected to govern the dynamics. To illustrate this concept, consider the double mass-spring-damper systems shown below. The ***differential equations*** of motion of these systems can be developed by applying Newton's laws to free-body diagrams of the individual masses.

The ***advantage*** of models that are based on differential equations is that they can be used to study the system dynamics under a ***variety*** of ***conditions***. The more accurately the equations model the physics of the system, the more useful the model results.

The ***form*** of the equations of motion depends on the ***specific configuration*** of the system. Double mass-spring-damper systems of various configurations are shown to demonstrate this fact.

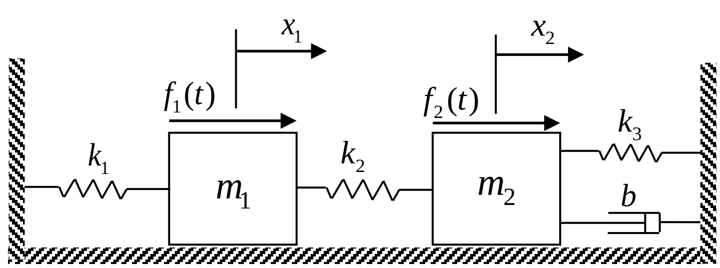
Double Mass-Spring-Damper Systems

1. For each of the following systems, find the two differential equations of motion. Assume the spring forces are proportional to the relative displacements of the masses and the damper forces are proportional to their relative velocities. The displacements $x_1(t)$ and $x_2(t)$ are the output for all the systems. Neglect friction.

   a. System #1: The forces $f_1(t)$ and $f_2(t)$ are the system input. The spring of stiffness k_1 is unstretched when $x_1 = 0$, the spring of stiffness k_2 is unstretched when $x_1 = x_2$, and the spring of stiffness k_3 is unstretched when $x_2 = 0$.

   Answer:

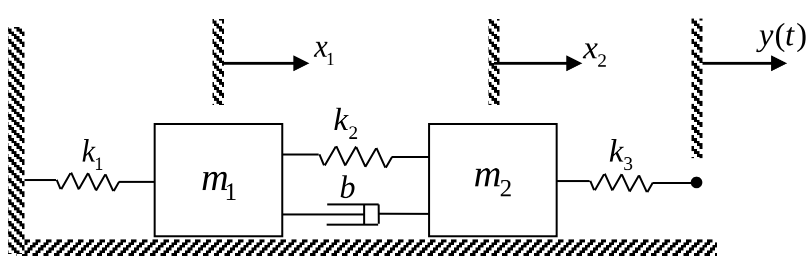
   $$\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 &= f_1(t) \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_2+k_3)x_2 - k_2x_1 &= f_2(t) \end{aligned}$$



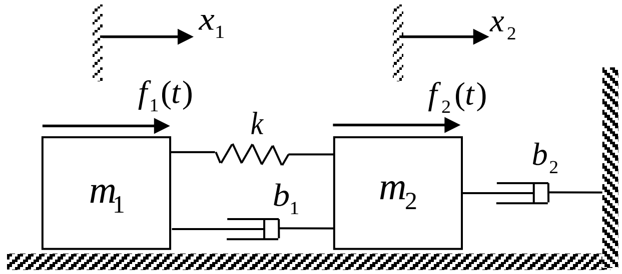
   b. System #2: The displacement $y(t)$ is the system input. The spring of stiffness k_1 is unstretched when $x_1 = 0$, the spring of stiffness k_2 is unstretched when $x_1 = x_2$, and the spring of stiffness k_3 is unstretched when $y = x_2$.

   Answer:

   $$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + (k_1+k_2)x_1 - b\dot{x}_2 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_2+k_3)x_2 - b\dot{x}_1 - k_2x_1 &= k_3y(t) \end{aligned}$$



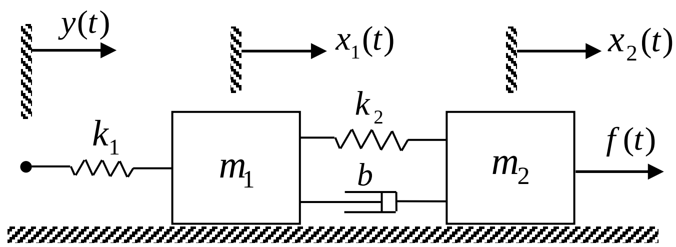
c. System #3: The forces $f_1(t)$ and $f_2(t)$ are the system input. The spring of stiffness k is unstretched when $x_1 = x_2$.



Answer:

$$
\begin{aligned}
m_1\ddot{x}_1 + b_1\dot{x}_1 + kx_1 - b_1\dot{x}_2 - kx_2 &= f_1(t) \\
m_2\ddot{x}_2 + \left(b_1 + b_2\right)\dot{x}_2 + kx_2 - b_1\dot{x}_1 - kx_1 &= f_2(t)
\end{aligned}
$$

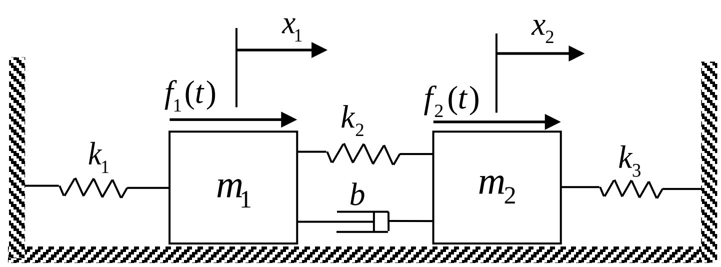
d. System #4: The displacement $y(t)$ and the force $f(t)$ are the system input. The spring of stiffness k_1 is unstretched when $x_1 = y$, and the spring of stiffness k_2 is unstretched when $x_1 = x_2$.



Answer:

$$
\begin{aligned}
m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 - b\dot{x}_2 - k_2x_2 &= k_1y(t) \\
m_2\ddot{x}_2 + b\dot{x}_2 + k_2x_2 - b\dot{x}_1 - k_2x_1 &= f(t)
\end{aligned}
$$

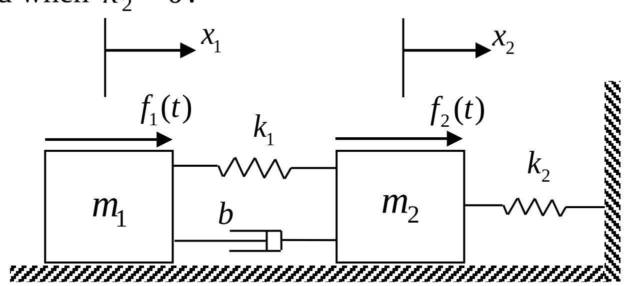
e. System #5: The forces $f_1(t)$ and $f_2(t)$ are the system input. The spring of stiffness k_1 is unstretched when $x_1 = 0$, the spring of stiffness k_2 is unstretched when $x_1 = x_2$, and the spring of stiffness k_3 is unstretched when $x_2 = 0$.



Answer:

$$
\begin{aligned}
m_1\ddot{x}_1 + b\dot{x}_1 + \left(k_1 + k_2\right)x_1 - b\dot{x}_2 - k_2x_2 &= f_1(t) \\
m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_2 + k_3\right)x_2 - b\dot{x}_1 - k_2x_1 &= f_2(t)
\end{aligned}
$$

f. System #6: The forces $f_1(t)$ and $f_2(t)$ are the system input. The spring of stiffness k_1 is unstretched when $x_1 = x_2$, and the spring of stiffness k_2 is unstretched when $x_2 = 0$.



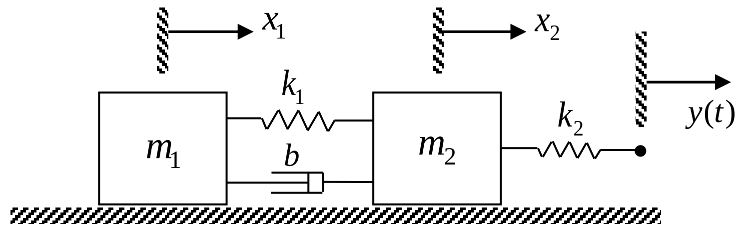
Answer:

$$
\begin{aligned}
m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 - b\dot{x}_2 - k_1x_2 &= f_1(t) \\
m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_1 + k_2\right)x_2 - b\dot{x}_1 - k_1x_1 &= f_2(t)
\end{aligned}
$$

g. System #7: The displacement $y(t)$ is the system input. The spring of stiffness k_1 is unstretched when $x_1 = x_2$, and the spring of stiffness k_2 is unstretched when $y = x_2$.

Answer:

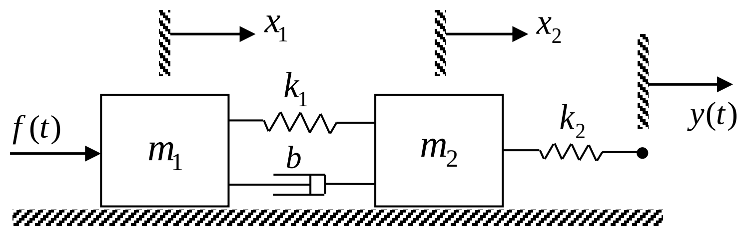
$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 - b\dot{x}_2 - k_1x_2 &= 0 \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_1+k_2)x_2 - b\dot{x}_1 - k_1x_1 &= k_2y(t) \end{aligned}$$



h. System #8: The displacement $y(t)$ and the force $f(t)$ are the system input. The spring of stiffness k_1 is unstretched when $x_1 = x_2$, and the spring of stiffness k_2 is unstretched when $x_2 = y$.

Answer:

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 - b\dot{x}_2 - k_1x_2 &= f(t) \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_1+k_2)x_2 - b\,\dot{x}_1 - k_1x_1 &= k_2y(t) \end{aligned}$$



Linearization of Mathematical Models

2. The model equations given below describe the output $y(t)$ of a system given its input $x(t)$. In each case, describe what makes the model equation nonlinear and find a linear approximate model to describe changes in the output associated with small deviations away from the stated input equilibrium value x_{eq}.

   a) Model equation: $y(t) = \dfrac{d^2x}{dt^2} + 5\big(\sin(x)\cos(x)\big)$; Equilibrium input value: $x_{eq} = 0$

      Answer: $\Delta y = \Delta\ddot{x} + 5\,\Delta x$

   b) Model equation: $y(t) = \dfrac{d^2x}{dt^2} + x^2$; Equilibrium input value: $x_{eq} = 2$

      Answer: $\Delta y = \Delta\ddot{x} + 4\,\Delta x$

   c) Model equation: $y(t) = 3\dfrac{d^3x}{dt^3} + 6\sqrt{x}$; Equilibrium input value: $x_{eq} = 9$.

      Can an *approximate linear model* be found for small deviations away from $x_{eq} = 0$?

      Answers: $\Delta y = 3\,\Delta\dddot{x} + \Delta x$. The linear model is undefined at $x_{eq} = 0$.

   d) Model equation: $y(t) = 5\dfrac{dx}{dt} + 6x + 6x^2$; Equilibrium input value: $x_{eq} = 2$

      Answer: $\Delta y = 5\Delta\dot{x} + 30\Delta x$

   e) Model equation: $y(t) = \dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + x^3$; Equilibrium input value: $x_{eq} = 2$

      Answer: $\Delta y = \Delta\ddot{x} + 3\Delta\dot{x} + 12\Delta x$

f) Model equation: $y(t) = 10\dfrac{dx}{dt} + x^3$; Equilibrium input value: $x_{eq} = 5$

Answer: $\Delta y = 10\Delta\dot{x} + 75\Delta x$

g) Model equation: $y(t) = 5\dfrac{d^2x}{dt^2} + 2x^3$; Equilibrium input value: $x_{eq} = 2$

Answer: $\Delta y = 5\Delta\ddot{x} + 24\Delta x$

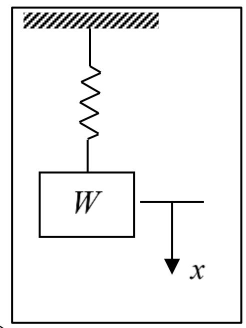
h) Model equation: $y = 5\dfrac{d^2x}{dt^2} + e^{2x}$; Equilibrium input value: $x_{eq} = 1$

Answer: $\Delta y = 5\Delta\ddot{x} + 14.78\Delta x$

i) Model equation: $y(t) = 4\dfrac{d^2x}{dt^2} + 12\sqrt{x}$; Equilibrium input value: $x_{eq} = 4$

Answer: $\Delta y = 4\Delta\ddot{x} + 3\Delta x$

3. The figure shows a block of weight W held in static equilibrium by a ***non-linear spring***. The force-displacement function $f(x)$ provides an estimate of the spring force for any displacement x (measured in feet) of the spring from its natural (or unstretched) length. In each of the cases below, find an approximate, linear model for the force-displacement function (FDF) associated with the given weight and equilibrium positions.



a) Weight: $W = 3$ (lb); Equilibrium Position: $x = x_{eq} = 0.5$ (ft); FDF: $f(x) = 5x + 2x^2$ (lb)

Answer: $\Delta f = 7\,\Delta x$ (lb)

b) Weight: $W = 5.25$ (lb); Equilibrium Position: $x = x_{eq} = 0.5$ (ft); FDF: $f(x) = 10x + 2x^3$ (lb)

Answer: $\Delta f = 11.5\,\Delta x$ (lb)

c) Weight: $W = 28$ (lb); Equilibrium Position: $x = x_{eq} = 2$ (ft); FDF: $f(x) = 10x + x^3$ (lb)

Answer: $\Delta f = 22\,\Delta x$ (lb)