Introductory Control Systems
Exercises #3 – Laplace Transforms and Ordinary Differential Equations

Ordinary Differential Equations

The ***dynamics*** of some physical systems are modeled by the following ***second-order***, ***ordinary differential equations***. Using ***Laplace transforms***, find the response of the system to the given input. In each case, assume the initial conditions are $x(0)=\dot{x}(0)=0$. Use partial fraction expansions as necessary. A table of common Laplace transforms is provided above in the notes. Identify the transient and steady-state parts of the response.

Note that, in general, Laplace transforms can be applied to systems of ***any order*** and to systems with ***nonzero*** initial conditions. Whereas the ***order*** of the differential equation generally ***changes*** from system to system, control system response is most often studied starting at ***zero initial conditions***.

1. $\ddot{x}+4\dot{x}+20x=f(t)$ with input $f(t)=100\left(1-e^{-5t}\right)$.

   Answer: $x(t)=\underbrace{5}_{\text{steady-state}}\underbrace{-4e^{-5t}-5.59e^{-2t}\sin(4t+0.1799)}_{\text{transient}}$

2. $\ddot{x}+4\dot{x}=f(t)$ with input $f(t)=130\,e^{-3t}\sin(2t)$.

   Answer: $x(t)=\underbrace{5}_{\text{steady-state}}\underbrace{-13e^{-4t}+16.1e^{-3t}\sin(2t+2.62)}_{\text{transient}}$

3. $\ddot{x}+5\dot{x}+6x=f(t)$ with input $f(t)=104\sin(2t)$.

   Answer: $x(t)=\underbrace{26e^{-2t}-16e^{-3t}}_{\text{transient}}\underbrace{-10.2\sin(2t+1.77)}_{\text{steady-state}}$

4. $\ddot{x}+6\dot{x}+13x=f(t)$ with input $f(t)=130\left(1-e^{-4t}\right)$.

   Answer: $x(t)=\underbrace{10}_{\text{steady-state}}\underbrace{-26e^{-4t}-32.25e^{-3t}\sin(2t+2.623)}_{\text{transient}}$

5. $\ddot{x}+8\dot{x}+12x=f(t)$ with input $f(t)=260\sin(4t)$.

   Answer: $x(t)=\underbrace{13e^{-2t}-5e^{-6t}}_{\text{transient}}-\underbrace{8.06\sin(4t+1.45)}_{\text{steady-state}}$

6. $\ddot{x}+4\dot{x}+25x=f(t)$ with input $f(t)=60\,e^{-5t}$.

   Answer: $x(t)=\underbrace{2e^{-5t}-2.39e^{-2t}\sin\left(\sqrt{21}\,t+2.15\right)}_{\text{transient}}$

7. $\ddot{x}+4\dot{x}+20x=f(t)$ with input $f(t)=100\left(1-e^{-5t}\right)$.

   Answer: $x(t)=\underbrace{5}_{\text{steady-state}}\underbrace{-4e^{-5t}-5.59e^{-2t}\sin\left(4t+0.1799\right)}_{\text{transient}}$

8. $\ddot{x}+6\dot{x}+16x=f(t)$ with input $f(t)=32\left(1-e^{-2t}\right)$.

Answer: $$x(t)=\underbrace{2}_{\text{steady-state}}\underbrace{-4e^{-2t}+2.138e^{-3t}\sin\left(\sqrt{7}\,t+1.932\right)}_{\text{transient}}$$

9. $\ddot{x}+6\dot{x}+8x=f(t)$ with input $f(t)=1189\cos(5t)$.

Answer: $$x(t)=\underbrace{-41e^{-2t}+58e^{-4t}}_{\text{transient}}\underbrace{-34.48\sin\left(5t+2.626\right)}_{\text{steady-state}}$$ or $$x(t)=\underbrace{-41e^{-2t}+58e^{-4t}}_{\text{transient}}\underbrace{-17\cos\left(5t\right)+30\sin\left(5t\right)}_{\text{steady-state}}$$

10. $\ddot{x}+4\dot{x}+40x=f(t)$ with input $f(t)=400e^{-10t}$.

Answer: $$x(t)=\underbrace{4e^{-10t}-\frac{40}{6}e^{-2t}\sin\left(6t+2.498\right)}_{\text{transient}}$$

11. $\ddot{x}+10\dot{x}+89x=f(t)$ with input $f(t)=136e^{-7t}$.

Answer: $$x(t)=\underbrace{2e^{-7t}-\frac{\sqrt{17}}{2}e^{-5t}\sin\left(8t+1.8158\right)}_{\text{transient}}$$

12. $\ddot{x}+8\dot{x}+12x=f(t)$ with input $f(t)=260\sin(4t)$.

Answer: $$x(t)=\underbrace{13e^{-2t}-5e^{-6t}}_{\text{transient}}\underbrace{-8.06\sin\left(4t+1.45\right)}_{\text{steady-state}}$$

13. $\ddot{x}+6\dot{x}+25x=f(t)$ with input $f(t)=425\left(1-e^{-2t}\right)$.

Answer: $$x(t)=\underbrace{17}_{\text{steady-state}}\underbrace{-25e^{-2t}+10.3e^{-3t}\sin\left(4t+2.25\right)}_{\text{transient}}$$

14. $\ddot{x}+9x=f(t)$ with input $f(t)=720\left(e^{-t}-e^{-3t}\right)$.

Answer: $$x(t)=\underbrace{72e^{-t}-40e^{-3t}}_{\text{transient}}\underbrace{-35.8\sin\left(3t+1.11\right)}_{\text{steady-state}}$$

15. $\ddot{x}+4\dot{x}=f(t)$ with input $f(t)=130\,e^{-3t}\sin(2t)$.

Answer: $$x(t)=\underbrace{5}_{\text{steady-state}}\underbrace{-13e^{-4t}-16.12e^{-3t}\sin\left(2t+2.618\right)}_{\text{transient}}$$

16. $\ddot{x}+5\dot{x}+4x=f(t)$ with input $f(t)=240\,e^{-3t}\cos(2t)$.

Answer: $$x(t)=\underbrace{20e^{-t}+16e^{-4t}-37.95e^{-3t}\sin\left(2t+1.249\right)}_{\text{transient}}$$

17. $\ddot{x}+4x=f(t)$ with input $f(t)=208\left(e^{-2t}-e^{-3t}\right)$.

Answer: $$x(t)=\underbrace{26e^{-2t}-16e^{-3t}}_{\text{transient}}\underbrace{-10.2\sin\left(2t+1.77\right)}_{\text{steady-state}}$$

18. $\ddot{x}+5\dot{x}+6x=f(t)$ with input $f(t)=25\sin(4t)$.

Answer: $x(t)=\underbrace{5e^{-2t}-4e^{-3t}}_{\text{transient}}\underbrace{-1.118\sin\left(4t+1.107\right)}_{\text{steady-state}}$ or $x(t)=\underbrace{5e^{-2t}-4e^{-3t}}_{\text{transient}}\underbrace{-\cos\left(4t\right)-0.5\sin\left(4t\right)}_{\text{steady-state}}$

Partial Fraction Expansions

For each of the systems below, identify the ***general form*** of the partial fraction expansion of $Y(s)$ and the ***general forms*** of the ***steady-state*** and ***transient*** parts of the inverse Laplace transform $y(t)$. Finally, using the ***final value theorem***, find $y_{ss}=\lim_{t\to\infty}(y(t))$ the steady-state value of $y(t)$.

Note that just knowing the ***form*** of the ***partial fraction expansion*** gives the analyst an idea of ***what kind*** of ***response*** to expect from the system.

1. $Y(s)=\dfrac{33(s^2+2s+8)}{s(s+3)(s^2+12s+100)}$

Answers: $Y(s)=\dfrac{K_1}{s}+\dfrac{K_2}{s+3}+\dfrac{As+B}{(s+6)^2+8^2}$; $y(t)=\underbrace{K_1}_{\text{steady-state}}+\underbrace{K_2e^{-3t}+K_3e^{-6t}\sin(8t+\phi)}_{\text{transient}}$; $y_{ss}=0.88$

2. $Y(s)=\dfrac{10(s^3+6s^2+2s+12)}{s(s+3)(s^2+16)}$

Answers: $Y(s)=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{Cs+D}{s^2+4^2}$; $y(t)=\underbrace{A}_{\text{steady-state}}+\underbrace{Be^{-3t}}_{\text{transient}}+\underbrace{E\sin(4t+\phi)}_{\text{steady-state}}$; No final value exists

3. $Y(s)=\dfrac{15(s^2+12s+20)}{(s+2)(s^2+9)(s^2+8s+52)}$

Answers: $Y(s)=\dfrac{K}{s+2}+\dfrac{As+B}{s^2+9}+\dfrac{Cs+D}{(s+4)^2+6^2}$; $y(t)=\underbrace{Ke^{-2t}+Ee^{-4t}\sin(6t+\phi_1)}_{\text{transient}}+\underbrace{F\sin(3t+\phi_2)}_{\text{steady-state}}$

No final value exists

4. $Y(s)=\dfrac{135(s^2+4s+10)}{s(s+2)(s^2+10s+30)(s^2+4s+25)}$

Answers: $Y(s)=\dfrac{A}{s}+\dfrac{B}{s+2}+\dfrac{Cs+D}{(s+5)^2+5}+\dfrac{Es+F}{(s+2)^2+21}$;

$y(t)=A+Be^{-2t}+Ge^{-5t}\sin\left(\sqrt{5}\,t+\phi_1\right)+He^{-2t}\sin\left(\sqrt{21}\,t+\phi_2\right)$; $y_{ss}=0.9$

5. $Y(s)=\dfrac{15(s^2+8s+12)}{(s+2)(s^2+16)(s^2+6s+45)}$

Answers: $Y(s)=\dfrac{K}{s+2}+\dfrac{As+B}{s^2+16}+\dfrac{Cs+D}{(s+3)^2+6^2}$; $y(t)=\underbrace{Ke^{-2t}}_{\text{over-damped}}+\underbrace{E\sin(4t+\phi_1)+Fe^{-3t}\sin(6t+\phi_2)}_{\text{under-damped}}$

No final value exists

6. $Y(s)=\dfrac{120(s+10)}{(s+2)(s^2+25)(s^2+6s+25)}$

Answers: $\boxed{Y(s)=\dfrac{K}{s+2}+\dfrac{As+B}{s^2+25}+\dfrac{Cs+D}{(s+3)^2+16}}$; $\boxed{y(t)=\underbrace{Ke^{-2t}}_{\text{over-damped}}+\underbrace{E\sin(5t+\phi_1)+Fe^{-3t}\sin(4t+\phi_2)}_{\text{under-damped}}}$

No final value exists

7. $Y(s)=\dfrac{4(s^3+4s^2+3s+12)}{(s+5)(s^2+16)(s^2+8s+25)}$

Answers: $\boxed{Y(s)=\dfrac{K}{s+5}+\dfrac{As+B}{s^2+16}+\dfrac{Cs+D}{(s+4)^2+9}}$; $\boxed{y(t)=\underbrace{Ke^{-5t}}_{\text{over-damped}}+\underbrace{E\sin(4t+\phi_1)+Fe^{-4t}\sin(3t+\phi_2)}_{\text{under-damped}}}$

No final value exists

8. $Y(s)=\dfrac{15(s^2+9s+21)}{(s+1)(s+6)(s^2+6s+25)}$

Answers: $\boxed{Y(s)=\dfrac{K_1}{s+1}+\dfrac{K_2}{s+6}+\dfrac{As+B}{(s+3)^2+4^2}}$; $\boxed{y(t)=\underbrace{K_1e^{-t}+K_2e^{-6t}}_{\text{over-damped}}+\underbrace{Ce^{-3t}\sin(4t+\phi)}_{\text{under-damped}}}$; $\boxed{y_{ss}=0}$

9. $Y(s)=\dfrac{4(s^2+7s+15)}{s(s+5)(s^2+6s+13)}$

Answers: $\boxed{Y(s)=\dfrac{K_1}{s}+\dfrac{K_2}{s+5}+\dfrac{As+B}{(s+3)^2+2^2}}$; $\boxed{y(t)=K_1+K_2e^{-5t}+K_3e^{-3t}\sin(2t+\phi)}$; $\boxed{y_{ss}=0.9231}$

10. $Y(s)=\dfrac{120(s+10)}{s(s^2+10s+74)(s^2+4s+20)}$

Answers: $\boxed{Y(s)=\dfrac{A}{s}+\dfrac{Bs+C}{(s+5)^2+49}+\dfrac{Ds+E}{(s+2)^2+16}}$; $\boxed{y(t)=A+Fe^{-5t}\sin\left(7t+\phi_1\right)+Ge^{-2t}\sin\left(4t+\phi_2\right)}$

$\boxed{y_{ss}=0.8108}$

11. $Y(s)=\dfrac{20(s^3+5s^2+20s+10)}{s(s+5)(s^2+8s+52)}$

Answers: $\boxed{Y(s)=\dfrac{A}{s}+\dfrac{B}{s+5}+\dfrac{Cs+D}{(s+4)^2+6^2}}$; $\boxed{y(t)=\underbrace{A}_{\text{steady-state}}+\underbrace{Be^{-5t}}_{\text{transient}}+\underbrace{Ee^{-4t}\sin(6t+\phi)}_{\text{transient}}}$; $\boxed{y_{ss}=0.7692}$

12. $Y(s)=\dfrac{4(s^2+10s+24)}{(s+3)(s^2+12s+52)}$

Answers: $\boxed{Y(s)=\dfrac{K}{s+3}+\dfrac{As+B}{(s+6)^2+4^2}}$; $\boxed{y(t)=\underbrace{Ke^{-3t}}_{\text{over-damped}}+\underbrace{Ce^{-6t}\sin(4t+\phi_1)}_{\text{under-damped}}}$; $\boxed{y_{ss}=0}$