Introductory Control Systems
Exercises #4 – Mathematical Models and Transfer Functions

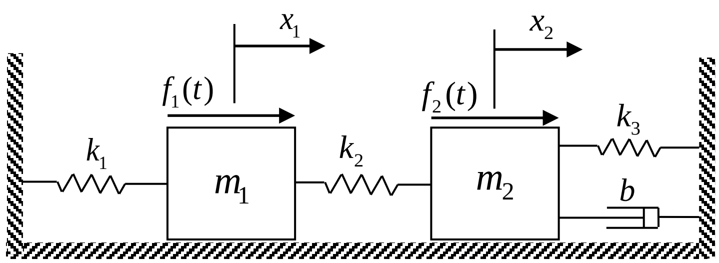
As previously stated, differential equations can be used to model system dynamics. Transfer functions associated with the system dynamics can be found by applying Laplace transforms to the differential equations. To illustrate this process a set of double mass-spring-damper systems are shown below. Find the indicated transfer function for each system by first applying Laplace transforms to the differential equations and then using Cramer's rule to solve for the indicated transfer function.

The differential equations of motion for each system were found as part of Exercises #2.

Double Mass-Spring-Damper Systems

a. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 &= f_1(t) \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_2+k_3)x_2 - k_2x_1 &= f_2(t) \end{aligned}$$

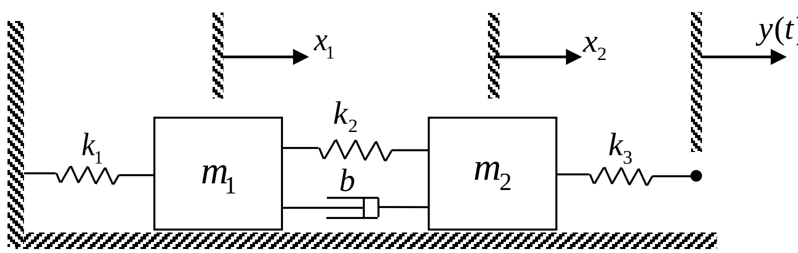


Find the transfer function $\dfrac{X_2}{F_1}(s)$.

Answer: $$\frac{X_2}{F_1}(s) = \frac{k_2}{(m_1s^2+k_1+k_2)(m_2s^2+bs+k_2+k_3)-k_2^2}$$

b. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + (k_1+k_2)x_1 - b\dot{x}_2 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_2+k_3)x_2 - b\dot{x}_1 - k_2x_1 &= k_3y(t) \end{aligned}$$

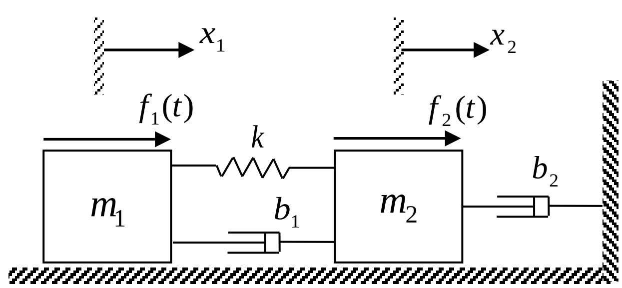


Find the transfer function $\dfrac{X_1}{Y}(s)$.

Answers: $$\frac{X_1}{Y}(s) = \frac{k_3(bs+k_2)}{(m_1s^2+bs+k_1+k_2)(m_2s^2+bs+k_2+k_3)-(bs+k_2)^2}$$

c. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + b_1\dot{x}_1 + k\,x_1 - b_1\dot{x}_2 - k\,x_2 &= f_1(t) \\ m_2\ddot{x}_2 + \left(b_1+b_2\right)\dot{x}_2 + k\,x_2 - b_1\dot{x}_1 - k\,x_1 &= f_2(t) \end{aligned}$$

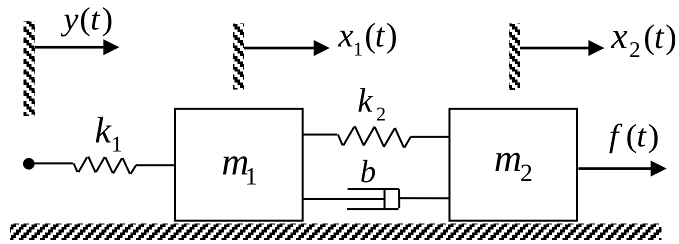


Find the transfer function $\dfrac{X_1}{F_2}(s)$.

Answers: $$\frac{X_1}{F_2}(s) = \frac{b_1s+k}{(m_1s^2+b_1s+k)\left(m_2s^2+(b_1+b_2)s+k\right)-(b_1s+k)^2}$$

d. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + (k_1+k_2)x_1 - b\dot{x}_2 - k_2x_2 &= k_1 y(t) \\ m_2\ddot{x}_2 + b\dot{x}_2 + k_2x_2 - b\dot{x}_1 - k_2x_1 &= f(t) \end{aligned}$$

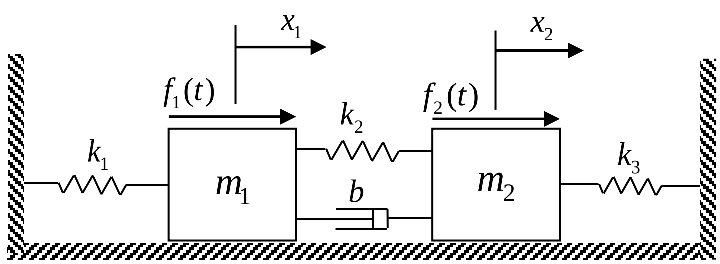


Find the transfer function $\frac{X_1}{F}(s)$.

Answers: $$\frac{X_1}{F}(s) = \frac{bs+k_2}{\left(m_1s^2+bs+k_1+k_2\right)\left(m_2s^2+bs+k_2\right)-(bs+k_2)^2}$$

e. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + \left(k_1+k_2\right)x_1 - b\dot{x}_2 - k_2x_2 &= f_1(t) \\ m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_2+k_3\right)x_2 - b\dot{x}_1 - k_2x_1 &= f_2(t) \end{aligned}$$

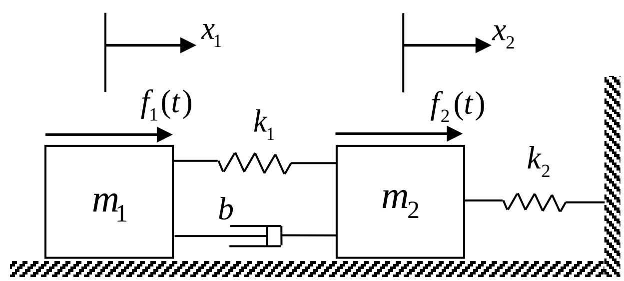


Find the transfer function $\frac{X_2}{F_1}(s)$.

Answers: $$\frac{X_2}{F_1}(s) = \frac{bs+k_2}{(m_1s^2+bs+k_1+k_2)\left(m_2s^2+bs+k_2+k_3\right)-(bs+k_2)^2}$$

f. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 - b\dot{x}_2 - k_1x_2 &= f_1(t) \\ m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_1+k_2\right)x_2 - b\dot{x}_1 - k_1x_1 &= f_2(t) \end{aligned}$$

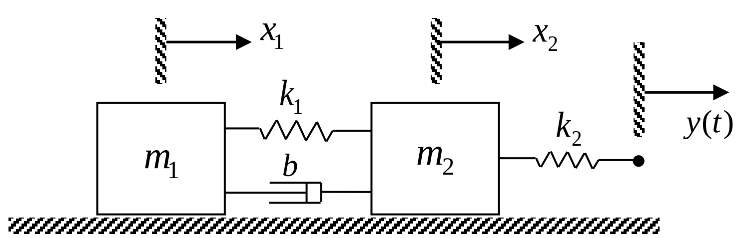


Find the transfer function $\frac{X_1}{F_2}(s)$.

Answers: $$\frac{X_1}{F_2}(s) = \frac{bs+k_1}{\left(m_1s^2+bs+k_1\right)\left(m_2s^2+bs+k_1+k_2\right)-\left(bs+k_1\right)^2}$$

g. The differential equations of motion of the system shown were previously found to be

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 - b\dot{x}_2 - k_1x_2 &= 0 \\ m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_1+k_2\right)x_2 - b\dot{x}_1 - k_1x_1 &= k_2 y(t) \end{aligned}$$

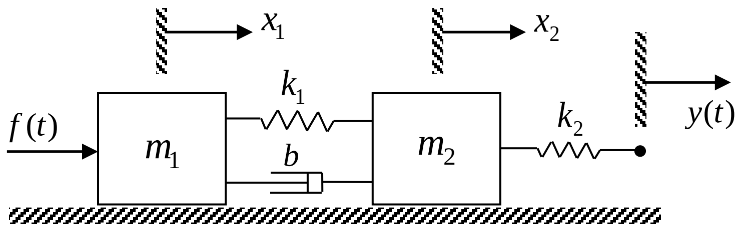


find the transfer function $\frac{X_2}{Y}(s)$.

Answers: $$\frac{X_2}{Y}(s) = \frac{k_2(m_1s^2+bs+k_1)}{(m_1s^2+bs+k_1)(m_2s^2+bs+k_1+k_2)-(bs+k_1)^2}$$

h. The differential equations of motion of the system shown were previously found to be

$$m_1\ddot{x}_1 + b\dot{x}_1 + k_1 x_1 - b\dot{x}_2 - k_1 x_2 = f(t)$$
$$m_2\ddot{x}_2 + b\dot{x}_2 + \left(k_1 + k_2\right) x_2 - b\,\dot{x}_1 - k_1 x_1 = k_2 y(t)$$



Find the transfer function $\dfrac{X_2}{F}(s)$.

Answers: $$\frac{X_2}{F}(s) = \frac{bs + k_1}{\left(m_1 s^2 + bs + k_1\right)\left(m_2 s^2 + bs + k_1 + k_2\right) - (bs + k_1)^2}$$