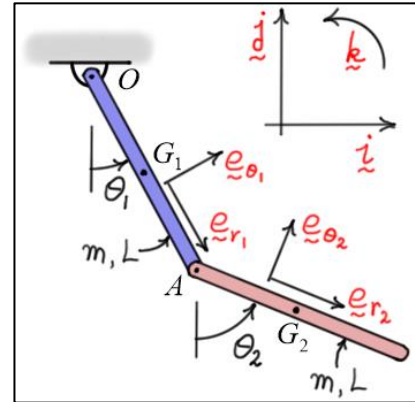


ME 6590 Multibody Dynamics

Examples for d'Alembert's Principle

Examples

1. Double Pendulum: Find the EOM of the double pendulum shown using d'Alembert's principle and θ_1 and θ_2 as the generalized coordinates. Both links have mass m and length L .



d'Alembert's Principle:

$$\sum_{i=1}^2 \left(m_i \underline{a}_{G_i} \cdot \frac{\partial \underline{v}_{G_i}}{\partial \dot{\theta}_k} \right) + \sum_{i=1}^2 \left[\left(\underline{I}_{G_i} \cdot \underline{\alpha}_{B_i} \right) + \left(\underline{\omega}_{B_i} \times \underline{H}_{G_i} \right) \right] \cdot \frac{\partial \underline{\omega}_{B_i}}{\partial \dot{\theta}_k} = F_{q_k} \quad (k=1, \dots, 2)$$

where

$$\begin{aligned} \underline{\omega}_{B_1} &= \dot{\theta}_1 \underline{k} & \frac{\partial \underline{\omega}_{B_1}}{\partial \dot{\theta}_1} &= \underline{k} & \frac{\partial \underline{\omega}_{B_1}}{\partial \dot{\theta}_2} &= 0 \\ \underline{\omega}_{B_2} &= \dot{\theta}_2 \underline{k} & \frac{\partial \underline{\omega}_{B_2}}{\partial \dot{\theta}_1} &= 0 & \frac{\partial \underline{\omega}_{B_2}}{\partial \dot{\theta}_2} &= \underline{k} \\ \underline{v}_{G_1} &= \frac{1}{2} L \dot{\theta}_1 \underline{e}_{\theta_1} & \frac{\partial \underline{v}_{G_1}}{\partial \dot{\theta}_1} &= \frac{1}{2} L \underline{e}_{\theta_1} & \frac{\partial \underline{v}_{G_1}}{\partial \dot{\theta}_2} &= 0 \\ \underline{v}_{G_2} &= L \dot{\theta}_1 \underline{e}_{\theta_1} + \frac{1}{2} L \dot{\theta}_2 \underline{e}_{\theta_2} & \frac{\partial \underline{v}_{G_2}}{\partial \dot{\theta}_1} &= L \underline{e}_{\theta_1} & \frac{\partial \underline{v}_{G_2}}{\partial \dot{\theta}_2} &= \frac{1}{2} L \underline{e}_{\theta_2} \end{aligned}$$

$$\underline{a}_{G_1} = \frac{1}{2} L \ddot{\theta}_1 \underline{e}_{\theta_1} - \frac{1}{2} L \dot{\theta}_1^2 \underline{e}_{r_1} \quad \underline{a}_{G_2} = \left(L \ddot{\theta}_1 \underline{e}_{\theta_1} - L \dot{\theta}_1^2 \underline{e}_{r_1} \right) + \left(\frac{1}{2} L \ddot{\theta}_2 \underline{e}_{\theta_2} - \frac{1}{2} L \dot{\theta}_2^2 \underline{e}_{r_2} \right)$$

$$m_1 \underline{a}_{G_1} \cdot \left(\frac{\partial \underline{v}_{G_1}}{\partial \dot{\theta}_1} \right) = \frac{1}{4} m L^2 \ddot{\theta}_1 \quad m_1 \underline{a}_{G_1} \cdot \left(\frac{\partial \underline{v}_{G_1}}{\partial \dot{\theta}_2} \right) = 0$$

$$m_2 \underline{a}_{G_2} \cdot \left(\frac{\partial \underline{v}_{G_2}}{\partial \dot{\theta}_1} \right) = m \left(L^2 \ddot{\theta}_1 + \frac{1}{2} L^2 \ddot{\theta}_2 C_{2-1} - \frac{1}{2} L^2 \dot{\theta}_2^2 S_{2-1} \right)$$

$$m_2 \underline{a}_{G_2} \cdot \left(\frac{\partial \underline{v}_{G_2}}{\partial \dot{\theta}_2} \right) = m \left(\frac{1}{4} L^2 \ddot{\theta}_2 + \frac{1}{2} L^2 \ddot{\theta}_1 C_{2-1} + \frac{1}{2} L^2 \dot{\theta}_1^2 S_{2-1} \right)$$

$$\underline{I}_{G_1} \cdot \underline{\alpha}_{B_1} = \left(\frac{1}{12} m L^2 \right) \ddot{\theta}_1 \underline{k} \quad \underline{I}_{G_2} \cdot \underline{\alpha}_{B_2} = \left(\frac{1}{12} m L^2 \right) \ddot{\theta}_2 \underline{k}$$

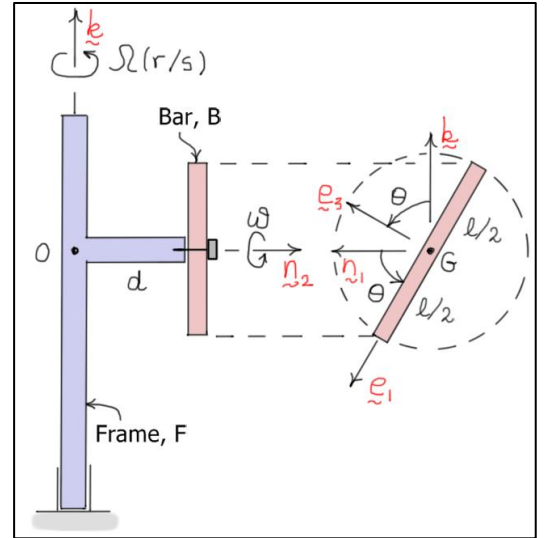
$$F_{\theta_1} = \left(-mg \underline{j} \right) \cdot \left(\frac{\partial \underline{v}_{G_1}}{\partial \dot{\theta}_1} \right) + \left(-mg \underline{j} \right) \cdot \left(\frac{\partial \underline{v}_{G_2}}{\partial \dot{\theta}_1} \right) = -\frac{3}{2} mg L S_1$$

$$F_{\theta_2} = \left(-mg \underline{j} \right) \cdot \left(\frac{\partial \underline{v}_{G_1}}{\partial \dot{\theta}_2} \right) + \left(-mg \underline{j} \right) \cdot \left(\frac{\partial \underline{v}_{G_2}}{\partial \dot{\theta}_2} \right) = -\frac{1}{2} mg L S_2$$

Substituting into d'Alembert's principle gives the two differential EOM

$$\begin{cases} \left(\frac{4}{3}mL^2\right)\ddot{\theta}_1 + \left(\frac{1}{2}mL^2C_{2-1}\right)\ddot{\theta}_2 - \left(\frac{1}{2}mL^2S_{2-1}\right)\dot{\theta}_2^2 + \left(\frac{3}{2}mgL\right)S_1 = 0 \\ \left(\frac{1}{2}mL^2C_{2-1}\right)\ddot{\theta}_1 + \left(\frac{1}{3}mL^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}mL^2S_{2-1}\right)\dot{\theta}_1^2 + \left(\frac{1}{2}mgL\right)S_2 = 0 \end{cases}$$

2. **Example System II (from ME 5550):** Find the *EOM* of the *bar* using d'Alembert's principle, given that the frame *F* is light (massless), the bar *B* has mass *m* and length *ℓ*, the motor torque $M_\theta(t)$ is applied between the frame and the bar, and that the motor torque $M_\phi(t)$ is applied between the ground and the frame *F*. Use ϕ ($\dot{\phi} = \Omega$) and θ ($\dot{\theta} = \omega$) as the generalized coordinates.



Reference frames: $F : (\underline{n}_1, \underline{n}_2, \underline{k})$, $B : (\underline{e}_1, \underline{n}_2, \underline{e}_3)$

d'Alembert's Principle:

$$\begin{cases} \left(m_B \underline{a}_{G_B} \cdot \frac{\partial \underline{v}_{G_B}}{\partial \dot{\phi}} \right) + \left[\left(\underline{I}_{G_B} \cdot \underline{\alpha}_B \right) + \left(\underline{\omega}_B \times \underline{H}_{G_B} \right) \right] \cdot \frac{\partial \underline{\omega}_B}{\partial \dot{\phi}} = F_\phi \\ \left(m_B \underline{a}_{G_B} \cdot \frac{\partial \underline{v}_{G_B}}{\partial \dot{\theta}} \right) + \left[\left(\underline{I}_{G_B} \cdot \underline{\alpha}_B \right) + \left(\underline{\omega}_B \times \underline{H}_{G_B} \right) \right] \cdot \frac{\partial \underline{\omega}_B}{\partial \dot{\theta}} = F_\theta \end{cases}$$

where

$$\begin{aligned} \underline{\omega}_F &= \dot{\phi} \underline{k} & \frac{\partial \underline{\omega}_F}{\partial \dot{\phi}} &= \underline{k} & \frac{\partial \underline{\omega}_F}{\partial \dot{\theta}} &= 0 \\ \underline{\omega}_B &= \dot{\theta} \underline{n}_2 + \dot{\phi} \underline{k} & \frac{\partial \underline{\omega}_B}{\partial \dot{\phi}} &= \underline{k} & \frac{\partial \underline{\omega}_B}{\partial \dot{\theta}} &= \underline{n}_2 = \underline{e}_2 \\ \underline{v}_{G_B} &= -d \dot{\phi} \underline{n}_1 & \frac{\partial \underline{v}_{G_B}}{\partial \dot{\phi}} &= -d \underline{n}_1 & \frac{\partial \underline{v}_{G_B}}{\partial \dot{\theta}} &= 0 \end{aligned}$$

$$\underline{a}_{G_B} = -d \ddot{\phi} \underline{n}_1 - d \dot{\phi}^2 \underline{n}_2$$

$$\underline{\alpha}_F = \ddot{\phi} \underline{k}$$

$$\underline{\alpha}_B = -\dot{\theta} \dot{\phi} \underline{n}_1 + \ddot{\theta} \underline{n}_2 + \ddot{\phi} \underline{k} = -(\ddot{\phi} S_\theta + \dot{\theta} \dot{\phi} C_\theta) \underline{e}_1 + \ddot{\theta} \underline{e}_2 + (\ddot{\phi} C_\theta - \dot{\theta} \dot{\phi} S_\theta) \underline{e}_3$$

$$m \underline{a}_{G_B} \cdot \left(\frac{\partial \underline{v}_{G_B}}{\partial \dot{\phi}} \right) = m d^2 \ddot{\phi}$$

$$m \underline{a}_{G_B} \cdot \left(\frac{\partial \underline{v}_{G_B}}{\partial \dot{\theta}} \right) = 0$$

$$\underline{H}_{G_B} = \frac{1}{12} m \ell^2 (\dot{\theta} \underline{e}_2 + \dot{\phi} C_\theta \underline{e}_3)$$

$$\underline{I}_{G_B} \cdot \underline{\alpha}_B = \frac{1}{12} m \ell^2 (\ddot{\theta} \underline{e}_2 + (\ddot{\phi} C_\theta - \dot{\theta} \dot{\phi} S_\theta) \underline{e}_3)$$

$$\underline{\omega}_B \times \underline{H}_G = \frac{1}{12} m \ell^2 \left((\dot{\phi}^2 S_\theta C_\theta) \underline{e}_2 + (-\dot{\theta} \dot{\phi} S_\theta) \underline{e}_3 \right)$$

$$\left(\underline{I}_{\underline{z}_G} \cdot \underline{\alpha}_B \right) \cdot \left(\partial \underline{\omega}_B / \partial \dot{\phi} \right) = \frac{1}{12} m \ell^2 \left(\ddot{\phi} C_\theta - \dot{\theta} \dot{\phi} S_\theta \right) C_\theta$$

$$\left(\underline{I}_{\underline{z}_G} \cdot \underline{\alpha}_B \right) \cdot \left(\partial \underline{\omega}_B / \partial \dot{\theta} \right) = \frac{1}{12} m \ell^2 \ddot{\theta}$$

$$\left(\underline{\omega}_B \times \underline{H}_G \right) \cdot \left(\partial \underline{\omega}_B / \partial \dot{\phi} \right) = -\frac{1}{12} m \ell^2 \dot{\theta} \dot{\phi} S_\theta C_\theta$$

$$\left(\underline{\omega}_B \times \underline{H}_G \right) \cdot \left(\partial \underline{\omega}_B / \partial \dot{\theta} \right) = \frac{1}{12} m \ell^2 \dot{\phi}^2 S_\theta C_\theta$$

$$F_\phi = \left(M_\theta \underline{n}_2 \right) \cdot \left(\partial \underline{\omega}_B / \partial \dot{\phi} \right) + \left(-M_\theta \underline{n}_2 \right) \cdot \left(\partial \underline{\omega}_F / \partial \dot{\phi} \right) + \left(M_\phi \underline{k} \right) \cdot \left(\partial \underline{\omega}_F / \partial \dot{\phi} \right) = M_\phi$$

$$F_\theta = \left(M_\theta \underline{n}_2 \right) \cdot \left(\partial \underline{\omega}_B / \partial \dot{\theta} \right) + \left(-M_\theta \underline{n}_2 \right) \cdot \left(\partial \underline{\omega}_F / \partial \dot{\theta} \right) + \left(M_\phi \underline{k} \right) \cdot \left(\partial \underline{\omega}_F / \partial \dot{\theta} \right) = M_\theta$$

Substituting into d'Alembert's principle give the two differential EOM

$$\boxed{\begin{aligned} \left(m d^2 + \frac{1}{12} m \ell^2 C_\theta^2 \right) \ddot{\phi} - \left(\frac{1}{6} m \ell^2 S_\theta C_\theta \right) \dot{\theta} \dot{\phi} &= M_\phi \\ \left(\frac{1}{12} m \ell^2 \right) \ddot{\theta} + \left(\frac{1}{12} m \ell^2 S_\theta C_\theta \right) \dot{\phi}^2 &= M_\theta \end{aligned}}$$

3. Single Link Pendulum with Constraints: Find the EOM of the pendulum using a dependent set of generalized coordinates and d'Alembert's principle with Lagrange multipliers. The bar has mass m and length ℓ and is driven by a motor torque M . The dependent set of generalized coordinates is $(q_1, q_2, q_3) = (x, y, \theta)$.

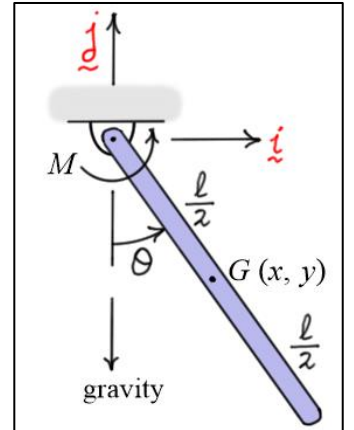
d'Alembert's Principle:

$$\boxed{m \underline{a}_G \cdot \frac{\partial \underline{v}_G}{\partial \dot{q}_k} + \left(\underline{I}_G \cdot \underline{\alpha} \right) \cdot \frac{\partial \underline{\omega}}{\partial \dot{q}_k} = F_{q_k} + \sum_{j=1}^2 \lambda_j a_{jk}} \quad (k=1, \dots, 2)$$

where

$$\begin{aligned} \underline{\omega} &= \dot{\theta} \underline{k} & \partial \underline{\omega} / \partial \dot{\theta} &= \underline{k} & \partial \underline{\omega} / \partial \dot{x} = \partial \underline{\omega} / \partial \dot{y} &= \underline{0} & \underline{\alpha} &= \ddot{\theta} \underline{k} \\ \underline{v}_G &= \dot{x} \underline{i} + \dot{y} \underline{j} & \partial \underline{v}_G / \partial \dot{x} &= \underline{i} & \partial \underline{v}_G / \partial \dot{y} &= \underline{j} & \partial \underline{v}_G / \partial \dot{\theta} &= \underline{0} \\ \underline{a}_G &= \ddot{x} \underline{i} + \ddot{y} \underline{j} & m \underline{a}_G \cdot \partial \underline{v}_G / \partial \dot{x} &= m \ddot{x} & m \underline{a}_G \cdot \partial \underline{v}_G / \partial \dot{y} &= m \ddot{y} & m \underline{a}_G \cdot \partial \underline{v}_G / \partial \dot{\theta} &= 0 \end{aligned}$$

$$\left(\underline{I}_{\underline{z}_G} \cdot \underline{\alpha} \right) \cdot \left(\partial \underline{\omega} / \partial \dot{\theta} \right) = \left(\frac{1}{12} m \ell^2 \right) \ddot{\theta}$$



$$F_\theta = -mg \underline{j} \cdot (\partial \underline{v}_G / \partial \dot{\theta}) + M \underline{k} \cdot (\partial \omega / \partial \dot{\theta}) = M$$

$$F_x = -mg \underline{j} \cdot (\partial \underline{v}_G / \partial \dot{x}) + M \underline{k} \cdot (\partial \omega / \partial \dot{x}) = 0$$

$$F_y = -mg \underline{j} \cdot (\partial \underline{v}_G / \partial \dot{y}) + M \underline{k} \cdot (\partial \omega / \partial \dot{y}) = -mg$$

Constraints:

$$x - \frac{\ell}{2} S_\theta = 0 \quad \text{or} \quad \dot{x} - \left(\frac{\ell}{2} C_\theta\right) \dot{\theta} = 0 \quad a_{11} = 1, \quad a_{12} = 0, \quad a_{13} = -\frac{\ell}{2} C_\theta$$

$$y + \frac{\ell}{2} C_\theta = 0 \quad \text{or} \quad \dot{y} - \left(\frac{\ell}{2} S_\theta\right) \dot{\theta} = 0 \quad a_{21} = 0, \quad a_{22} = 1, \quad a_{23} = -\frac{\ell}{2} S_\theta$$

Substituting into d'Alembert's principle give the three differential/algebraic EOM

$$\begin{array}{l} m \ddot{x} = \lambda_1 \\ m \ddot{y} = -mg + \lambda_2 \\ \left(\frac{1}{12} m \ell^2\right) \ddot{\theta} = M - \left(\frac{\ell}{2} C_\theta\right) \lambda_1 - \left(\frac{\ell}{2} S_\theta\right) \lambda_2 \end{array}$$

The two differentiated constraint equations are

$$\begin{array}{l} \ddot{x} - \left(\frac{\ell}{2} C_\theta\right) \ddot{\theta} + \left(\frac{\ell}{2} S_\theta\right) \dot{\theta}^2 = 0 \\ \ddot{y} - \left(\frac{\ell}{2} S_\theta\right) \ddot{\theta} - \left(\frac{\ell}{2} C_\theta\right) \dot{\theta}^2 = 0 \end{array}$$

Together, these five equations can be solved for x , y , θ , λ_1 , and λ_2 .