

ME 6590 Multibody Dynamics

d'Alembert's Principle for Multi-Degree-of-Freedom (MDOF) Systems

- Consider a system of “ N_B ” rigid bodies with “ n ” degrees of freedom (DOF). Previously, we noted that the equations of motion (EOM) of a such a system can be written for the “ n ” generalized coordinates q_k ($k=1,\dots,n$) using Lagrange's equations. Another approach to finding the EOM is to use **d'Alembert's principle**. For a system of “ N_B ” rigid bodies with “ n ” DOF, d'Alembert's principle may be written

$$\sum_{i=1}^{N_B} \left(m_i \underline{a}_{G_i} \cdot \frac{\partial \underline{v}_{G_i}}{\partial \dot{q}_k} \right) + \sum_{i=1}^{N_B} \left[\left(\underline{I}_{G_i} \cdot \underline{\alpha}_{B_i} \right) + \left(\underline{\omega}_{B_i} \times \underline{H}_{G_i} \right) \right] \cdot \frac{\partial \underline{\omega}_{B_i}}{\partial \dot{q}_k} = F_{q_k} \quad (k=1,\dots,n) \quad (1)$$

where

m_{B_i} = mass of body B_i

\underline{v}_{G_i} = velocity of G_i , the mass center of B_i

\underline{a}_{G_i} = acceleration of G_i , the mass center of B_i

$\underline{\omega}_{B_i}$ = angular velocity of B_i

$\underline{\alpha}_{B_i}$ = angular acceleration of B_i

\underline{I}_{G_i} = inertia dyadic for B_i

\underline{H}_{G_i} = angular momentum of B_i about its mass center G_i

F_{q_k} = generalized force associated with generalized coordinate q_k

(due to all forces and torques acting on the system)

Notes

1. Eqs. (1) represent “ n ” differential equations for the “ n ” generalized coordinates q_k ($k=1,\dots,n$). It is important that all quantities be written only in terms of q_k , \dot{q}_k , and no other variables.
2. The **right-hand-side** of the EOM are the **same** as for Lagrange's equations. All forces and torques (both conservative and nonconservative) are included in F_{q_k} . There is no special form for conservative systems.
3. This form of d'Alembert's principle can be used for systems **without constraints** or for systems with **holonomic constraints** provided it is easy to use the constraints to **eliminate surplus generalized coordinates**.

4. For systems with *nonholonomic constraints* or for systems with *holonomic constraints* for which it is not convenient to eliminate surplus generalized coordinates, then *Lagrange multipliers* are used.

d'Alembert's Principle with Constraints

- If the configuration of a dynamic system is to be described using “ n ” *generalized coordinates* q_k ($k = 1, \dots, n$), and if the system is subjected to “ m ” *independent* holonomic and/or nonholonomic *constraints* of the form

$$\boxed{\sum_{k=1}^n a_{jk} \dot{q}_k + a_{j0} = 0} \quad (j = 1, \dots, m) \quad (2)$$

then the system possesses “ $N = n - m$ ” DOF. In this case, the EOM of the system may be formulated using d'Alembert's principle with Lagrange multipliers which can be written as follows.

$$\boxed{\sum_{i=1}^{N_B} \left(m_i a_{G_i} \cdot \frac{\partial v_{G_i}}{\partial \dot{q}_k} \right) + \sum_{i=1}^{N_B} \left[\left(\underline{I}_{G_i} \cdot \underline{\alpha}_{B_i} \right) + \left(\underline{\omega}_{B_i} \times \underline{H}_{G_i} \right) \right] \cdot \frac{\partial \underline{\omega}_{B_i}}{\partial \dot{q}_k} = F_{q_k} + \sum_{j=1}^m \lambda_j a_{jk}} \quad (k = 1, \dots, n) \quad (3)$$

- Here, λ_j is the *Lagrange multiplier* associated with the j^{th} constraint equation, a_{jk} ($j = 1, \dots, m; k = 1, \dots, n$) are the *coefficients* from the constraint equations, and all other quantities are as defined above.
- The “ n ” d'Alembert's principle equations (3) and the “ m ” constraint equations (2) form a set of “ $n + m$ ” *differential/algebraic* equations for the “ $n + m$ ” unknowns – the “ n ” generalized coordinates q_k ($k = 1, \dots, n$) and the “ m ” Lagrange multipliers λ_j ($j = 1, \dots, m$).
- The equations are *differential* in the *generalized coordinates* and *algebraic* in the *Lagrange multipliers*. The Lagrange multipliers are related to the forces and moments required to maintain the constraints. Note that, it is important that all quantities be written only in terms of q_k , \dot{q}_k , and no other variables.