ME 6590 Multibody Dynamics
Kane's Equations for Multi-Degree-of-Freedom (MDOF) Systems

- Consider a system of "N_B" rigid bodies with "n" degrees of freedom (DOF). Previously, it was noted that equations of motion (EOM) of a such a system can be written for the "n" generalized coordinates $q_k \; (k=1,\ldots,n)$ using Lagrange's equations or d'Alembert's principle. Another approach is to use ***Kane's Equations***. For a system of "N_B" rigid bodies with "n" DOF, Kane's Equations may be written as

$$\boxed{\sum_{i=1}^{N_B}\left(m_i \, \underset{\sim}{a}_{G_i} \cdot \frac{\partial \underset{\sim}{v}_{G_i}}{\partial u_k}\right) + \sum_{i=1}^{N_B}\left[\left(\underset{\approx}{I}_{G_i} \cdot \underset{\sim}{\alpha}_{B_i}\right) + \left(\underset{\sim}{\omega}_{B_i} \times \underset{\sim}{H}_{G_i}\right)\right] \cdot \frac{\partial \underset{\sim}{\omega}_{B_i}}{\partial u_k} = F_{u_k}} \qquad (k=1,\ldots,n) \qquad (1)$$

where

$$\begin{aligned}
m_{B_i} &= \text{mass of body } B_i \\
\underset{\sim}{v}_{G_i} &= \text{velocity of } G_i\text{, the mass center of } B_i \\
\underset{\sim}{a}_{G_i} &= \text{acceleration of } G_i\text{, the mass center of } B_i \\
\underset{\sim}{\omega}_{B_i} &= \text{angular velocity of } B_i \\
\underset{\sim}{\alpha}_{B_i} &= \text{angular acceleration of } B_i \\
\underset{\approx}{I}_{G_i} &= \text{inertia dyadic for } B_i \\
\underset{\sim}{H}_{G_i} &= \text{angular momentum of } B_i \text{ about its mass center } G_i \\
F_{u_k} &= \sum_j \left(\underset{\sim}{F}_j \cdot \partial \underset{\sim}{v}_{P_j} \big/ \partial u_k\right) + \sum_j \left(\underset{\sim}{M}_j \cdot \partial \underset{\sim}{\omega}_{B_j} \big/ \partial u_k\right) \\
&= \text{generalized force associated with generalized speed } u_k \\
&\qquad \text{(due to all forces and torques acting on the system)}
\end{aligned} \qquad (2)$$

Here, $u_k \; (k=1,\ldots,n)$ are an ***independent*** set of ***generalized speeds***, and the generalized speeds can be ***linear combinations*** of the generalized coordinate derivatives.

<u>Notes</u>

1. Eqs. (1) along with the ***kinematic differential equations*** represent "$2n$" first-order differential equations for the "n" generalized coordinates $q_k \; (k=1,\ldots,n)$ and the "n" generalized speeds $u_k \; (k=1,\ldots,n)$. It is important that all quantities be written only in terms of q_k, u_k, and no other variables.

2. The right-hand-side of the EOM are ***similar*** to that for Lagrange's equations and d'Alembert's principle, except that the ***partial velocities*** and ***partial angular velocities*** are defined for the ***generalized speeds***. The generalized speeds can be the derivatives of the generalized coordinates, but they can also be linear combinations of the derivatives of the generalized coordinates. All forces and torques (both conservative and nonconservative) are included in F_{u_k} .
3. ***Kane's equations*** are ***very similar*** to ***d'Alembert's principle***, except they are based on selecting an ***independent*** set of ***generalized speeds***. d'Alembert's principle is based on selecting an ***independent*** set of ***generalized coordinates***.
4. As presented above, Kane's equations can also be written using a set of ***independent*** generalized speeds u_k $(k=1,\ldots,n)$ and a set of ***dependent*** generalized coordinates. Note that only the generalized speeds must be independent. In this case, Kane's equations must then be supplemented with ***kinematical differential equations*** and ***differentiated constraint equations*** (relating the dependent generalized coordinates) so the total number of first-order differential equations is equal to the total number of ***independent generalized speeds plus the total number of generalized coordinates***. This characteristic makes Kane's equations more versatile than d'Alembert's principle.
5. If a ***dependent set of generalized speeds*** are selected, then Lagrange multipliers are used as described below.

<u>Kane's Equations with Dependent Generalized Speeds</u>

- If the configuration of a dynamic system is to be described using "*n*" ***generalized coordinates*** q_k $(k=1,\ldots,n)$ and if the system is subject to "*m*" ***independent*** holonomic and/or nonholonomic ***constraints*** of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}u_k + a_{j0} \ = \ 0} \qquad (j=1,\ldots,m) \tag{3}$$

then the system possesses $N = n - m$ DOF.

- If a set of ***dependent set of generalized speeds*** are used, then the equations of motion can be formulated using Kane's equations with ***Lagrange multipliers*** as follows.

$$\boxed{\sum_{i=1}^{N_B}\left(m_i\, \underset{\sim}{a}_{G_i}\cdot\frac{\partial \underset{\sim}{v}_{G_i}}{\partial u_k}\right)+\sum_{i=1}^{N_B}\left[\left(\underset{\approx}{I}_{G_i}\cdot\underset{\sim}{\alpha}_{B_i}\right)+\left(\underset{\sim}{\omega}_{B_i}\times\underset{\sim}{H}_{G_i}\right)\right]\cdot\frac{\partial \underset{\sim}{\omega}_{B_i}}{\partial u_k}=F_{u_k}+\sum_{j=1}^{m}\lambda_j a_{jk}}\;(k=1,\ldots,n) \quad (4)$$

- Here, λ_j is the Lagrange multiplier associated with the j^{th} constraint equation, a_{jk} $(j=1,\ldots,m;\ k=1,\ldots,n)$ are the coefficients from the constraint equations, and all other quantities are as defined above.
- The "n" Kane's equations (4), the "n" kinematic differential equations, and the "m" constraint equations (3) form a set of "$2n+m$" ***differential/algebraic equations*** for the "$2n+m$" unknowns – the "n" generalized coordinates q_k $(k=1,\ldots,n)$, the "n" generalized speeds u_k $(k=1,\ldots,n)$, and the "m" Lagrange multipliers λ_j $(j=1,\ldots,m)$.
- As before, the Lagrange multipliers are related to the forces and moments required to maintain the constraints. Note that, it is important that all quantities be written only in terms of q_k, u_k, and no other variables.