Summary

This unit discusses the application of Lagrange’s equations, d’Alembert’s principle and Kane’s equations to rigid body dynamic systems with constraints. In Units 4 and 5 of this volume, the applications of Lagrange’s equations and d’Alembert’s principle were based on a set of independent generalized coordinates, and the applications of Kane’s equations were based on a set of independent generalized speeds. There are systems for which it is inconvenient or impossible to eliminate surplus generalized coordinates or simply inconvenient to eliminate generalized speeds from the analysis. For these systems, the application of each of these three methods can be supplemented with the use of Lagrange multipliers.

The resulting equations of motion are a set of differential and algebraic equations. The Lagrange multipliers are algebraic unknowns related to the forces and torques required to maintain the constraints. An Addendum to this unit explores this connection.

In the analysis that follows, constraint equations are assumed to be equality constraints, that is, some function of the generalized coordinates and/or generalized speeds is equal to zero. The case of inequality constraints is not considered.
Configuration Constraints for Mechanical Systems

Suppose the configuration of a mechanical system is defined by “n” generalized coordinates, say \( q_k \) \((k = 1, \ldots, n)\). These coordinates may all be independent, or they may form a dependent set. For example, consider the single link pendulum. The coordinate set \( \{x_G, y_G, \theta\} \) is a dependent set of coordinates, because the following independent constraint equations can be used to relate the coordinates

\[
\begin{align*}
x_G &= \frac{1}{2} L \sin(\theta) \\
y_G &= \frac{1}{2} L \cos(\theta)
\end{align*}
\]

Hence, for this system only one generalized coordinate is required. Any set of two or more coordinates for this system forms a dependent set. Note that care must be taken to include only independent constraint equations in this process.

The types of constraints described above are referred to as configuration constraints. They directly relate some or all the generalized coordinates. For a mechanical system described by “n” generalized coordinates \( q_k \) \((k = 1, \ldots, n)\) with “m” independent configuration constraints, many constraints can be written in the form

\[
f_j(q_1, q_2, \ldots, q_n, t) = 0 \quad (j = 1, \ldots, m)
\]

Here, the functions \( f_j \) \((j = 1, \ldots, m)\) and their derivatives through second order with respect to all variables are assumed to be continuous.

Configuration constraints are most useful in a dynamic analysis when they are differentiated into a form linear in the time derivatives of the generalized coordinates. Using concepts from multivariate calculus, the time derivatives of the constraint functions \( f_j \) can be written as

\[
\frac{df_j}{dt} = \sum_{k=1}^{n} \left( \frac{\partial f_j}{\partial q_k} \right) \dot{q}_k + \frac{\partial f_j}{\partial t} = 0 \quad \text{or} \quad \sum_{k=1}^{n} a_{jk} \dot{q}_k + a_{j0} = 0 \quad (j = 1, \ldots, m)
\]

Eq. (2) can be written in matrix form as

\[
\begin{bmatrix} A_{m \times n} \end{bmatrix} \begin{bmatrix} \dot{q} \end{bmatrix}_{n \times 1} + \begin{bmatrix} a_0 \end{bmatrix}_{m \times 1} = \begin{bmatrix} 0 \end{bmatrix}_{m \times 1}
\]

(3)

By comparing these expressions, the elements of the coefficient matrix \( A_{m \times n} \) and the column vector \( a_0_{m \times 1} \) are identified to be

\[
a_{jk} \triangleq \frac{\partial f_j}{\partial q_k} \quad (j = 1, \ldots, m; \ k = 1, \ldots, n) \quad \text{and} \quad a_{j0} \triangleq \frac{\partial f_j}{\partial t} \quad (j = 1, \ldots, m)
\]

(4)
In this form, the \textit{independence} of the constraint equations can be \textit{verified} by checking the \textit{rank} of the coefficient matrix $[A]$. If the rank of $[A]$ is “$m$”, then the equations are \textit{independent}. If the rank of $[A]$ is less than “$m$”, then the equations are \textit{dependent}. The rank of $[A]$ is equal to the number of independent constraint equations.

Configuration constraints are called \textit{holonomic} constraints. If they \textit{depend} explicitly on the time, they are called \textit{rheonomic} constraints. If they \textit{do not depend} explicitly on the time, they are called \textit{schleronomic} constraints.

\textbf{Examples}

1. If (as described above) the generalized coordinate set $\{q_1, q_2, q_3\} = \{x_G, y_G, \theta\}$ is used to describe the configuration of the single link pendulum, then two configuration constraints can be written as

$$f_1(q_1, q_2, q_3) = q_1 - \frac{1}{2} L \sin(q_3) = 0$$
$$f_2(q_1, q_2, q_3) = q_2 - \frac{1}{2} L \cos(q_3) = 0$$

Differentiating these equations with respect to time using the \textit{chain rule} gives

$$\begin{bmatrix} \dot{q}_1 - \frac{1}{2} L \dot{q}_3 \cos(q_3) \\ \dot{q}_2 + \frac{1}{2} L \dot{q}_3 \sin(q_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} L \cos(q_3) \\ 0 & 1 & +\frac{1}{2} L \sin(q_3) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \equiv [A] \{\dot{q}\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Comparing these results to the general form shown above gives

$$a_{11} = a_{22} = 1 \quad a_{12} = a_{21} = 0 \quad a_{13} = -\frac{1}{2} L \cos(\theta) \quad a_{23} = \frac{1}{2} L \sin(\theta) \quad a_{10} = 0 \quad (j = 1, 2)$$

2. If the generalized coordinate set $\{q_1, q_2\} = \{x_G, y_G\}$ is used to describe the configuration of the single link pendulum, then there is only one independent configuration constraint. For example,

$$f(q_1, q_2) = q_1^2 + q_2^2 - \left(\frac{1}{2} L\right)^2 = 0$$

Differentiating this equation with respect to time using the \textit{chain rule} gives

$$2q_1\dot{q}_1 + 2q_2\dot{q}_2 = 0 \quad \Rightarrow \quad \{q_1 \dot{q}_1, q_2 \dot{q}_2\} = 0 \quad \Rightarrow \quad [q_1 \ q_2] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \equiv [A] \{\dot{q}\} = 0$$

So, in this case, $a_{11} = q_1 = x_G$, $a_{12} = q_2 = y_G$, and $a_{10} = 0$. 

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Motion Constraints for Mechanical Systems

Configuration constraints directly relate some or all the coordinates used to describe the configuration of a system. The basic form of a configuration constraint is as shown in Eq. (1). In contrast, motion constraints directly relate some or all of the derivatives of the generalized coordinates as presented in Eq. (2). If the motion constraint can be integrated, an equivalent configuration constraint in the form of Eq. (1) can be found. Configuration constraints and motion constraints that can be integrated are called holonomic constraints. Motion constraints that cannot be integrated are called nonholonomic constraints.

Given that the constraint functions \( f_j \ (j=1,\ldots,m) \) and their derivatives through second order with respect to all variables are continuous, not only is Eq. (4) true, but all the mixed second partial derivatives must be equal. That is,

\[
\frac{\partial a_{jk}}{\partial q_{\ell}} = \frac{\partial^2 f_j}{\partial q_k \partial q_{\ell}} = \frac{\partial^2 f_j}{\partial q_k \partial q_{\ell}} = \frac{\partial a_{\ell j}}{\partial q_k} \quad k, \ell = (1, \ldots, n)
\]

and

\[
\frac{\partial a_{j0}}{\partial q_{\ell}} = \frac{\partial^2 f_j}{\partial q_{\ell} \partial \dot{q}_t} = \frac{\partial^2 f_j}{\partial q_{\ell} \partial \dot{q}_t} = \frac{\partial a_{\ell j}}{\partial q_{\ell}} \quad \ell = (1, \ldots, n)
\]

If these equations are satisfied for a given motion constraint, then the constraint is integrable and the constraint is holonomic. Unfortunately, if the above equations are not satisfied, it cannot be immediately concluded that the constraint is nonholonomic, because it is possible that an integrating factor exists that transforms the equation into an exact differential. Unfortunately, finding an integrating factor is not always an easy task. In some (usually simple) cases they are not difficult to find, but in many cases the analyst is left only with a trial and error process.

Equations of Motion for Systems with Constraints

The forms of Lagrange’s equations and d’Alembert’s principle presented in Units 4 and 5 of this volume require that a set of independent generalized coordinates be used. The form of Kane’s equations presented in Unit 5 of this volume allows for a set of dependent generalized coordinates but requires a set of independent generalized speeds. The use of independent generalized coordinates in Lagrange’s equations or d’Alembert’s principle or the use of independent generalized speeds in Kane’s equations is not always practical or possible.

In the case of configuration constraints or integrable motion constraints (i.e. holonomic constraints), it is theoretically possible to eliminate excess generalized coordinates from the analysis by solving the constraint equations. However, this process may provide equations of motion that are overly complicated and very tedious to generate. In the case of non-integrable motion constraints (i.e. nonholonomic constraints), it is not possible to eliminate excess generalized coordinates from the analysis. It is, however, possible to eliminate...
the excess generalized speeds. But, as with the elimination of excess generalized coordinates, the process of eliminating excess generalized speeds may provide equations of motion that are overly complicated and very tedious to generate.

Lagrange’s Equations and d’Alembert’s Principle for a Set of Dependent Generalized Coordinates

For the reasons mentioned above, it is often advantageous or necessary to use a set of dependent generalized coordinates \( q_k \) \((k = 1, \ldots, n)\) to describe the configuration of a mechanical system. If the system possesses \( N = n - m \) degrees of freedom (DOF), then there are \( m \) independent constraint equations that can be written in the form of Eq. (2) or Eq. (3) above. In this case, Lagrange’s equations and d’Alembert’s principle can be modified using a set of unknown Lagrange multipliers. The specific forms of the equations follow.

Lagrange’s Equations:

\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k = 1, \ldots, n)
\]

(5)

or

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \left( F_{q_k} \right)_{nc} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k = 1, \ldots, n)
\]

(6)

d’Alembert’s Principle:

\[
\sum_{i=1}^{N} \left( m_i r G_i \cdot \frac{\partial \omega_i}{\partial \dot{q}_k} \right) + \sum_{i=1}^{N} \left[ \left( l G_i \cdot r a_{ji} \right) + \left( r \omega_i \times H G_i \right) \right] \cdot \frac{\partial \omega_k}{\partial \dot{q}_k} = F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k = 1, \ldots, n)
\]

(7)

These equations are identical to those presented in Units 4 and 5 of this volume except for the summation term \( \sum_{j=1}^{m} \lambda_j a_{jk} \) that has been added to the right side of the equations. The variables \( \lambda_j \) \((j = 1, \ldots, m)\) are a set of unknown Lagrange multipliers, one for each constraint equation, and the variables \( a_{jk} \) \((j = 1, \ldots, m; k = 1, \ldots, n)\) are the coefficients from the constraint equations expressed in the form of Eq. (2). Clearly, the Lagrange multipliers appear in the equations as algebraic unknowns.

The \( n \) equations (from Lagrange’s equations or d’Alembert’s principle) and the \( m \) constraint equations form a set of \( n + m \) differential/algebraic equations for the \( n + m \) unknowns – the \( n \) generalized coordinates \( q_k \) \((k = 1, \ldots, n)\) and the \( m \) Lagrange multipliers \( \lambda_j \) \((j = 1, \ldots, m)\). The Lagrange multipliers can be related to the forces and moments required to maintain the constraints. (See the Addendum at the end of this Unit.) It is important to note that all terms in the equations be written only in terms of \( q_k \) \((k = 1, \ldots, n)\) and their derivatives and no other variables.
Kane’s Equations for a Set of Dependent Generalized Speeds

For the reasons mentioned above, it may be advantageous to use a set of dependent generalized speeds $u_k$ ($k=1,\ldots,n$). If the system possesses $N=n-m$ degrees of freedom, and there are $m$ independent constraint equations that can be written of the form

$$\sum_{k=1}^{n} a_{jk} u_k + a_{j0} = 0 \quad (j=1,\ldots,m) \quad (8)$$

then, Kane’s equations can be modified using a set of unknown Lagrange multipliers. In this case, Kane’s equations can be written as

$$\sum_{i=1}^{N_c} \left( m \frac{R}{\partial q_{i}} \frac{\partial R}{\partial q_{G_{i}}} \right) + \sum_{i=1}^{N_c} \left[ \left( \frac{R}{\partial q_{B_{i}}} \frac{R}{\partial q_{B_{i}}} \right) + \left( \frac{R}{\partial q_{B_{i}}} \frac{R}{\partial q_{B_{i}}} \right) \right] \frac{\partial R}{\partial u_k} = F_k + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k=1,\ldots,n) \quad (9)$$

If the system configuration is described by the generalized coordinates $q_k$ ($k=1,\ldots,n_c$), then the $n$ Kane’s equations along with the $n_c$ kinematic differential equations and the $m$ constraint equations form a set of $n+n_c+m$ differential and algebraic equations in as many unknowns – the $n_c$ generalized coordinates, the $n$ generalized speeds $u_k$ ($k=1,\ldots,n$), and the $m$ Lagrange multipliers $\lambda_j$ ($j=1,\ldots,m$). As before, the Lagrange multipliers can be related to the forces and moments required to maintain the constraints. Note that, it is important that all quantities be written only in terms of $q_k$ ($k=1,\ldots,n_c$), $u_k$ ($k=1,\ldots,n$) and their derivatives and no other variables.

**Example 1: Equations of Motion of the Simple Planar Pendulum**

The diagram shows a simple pendulum – a point mass (mass, $m$ ) on the end of a light rod of length $L$. The X-axis is horizontal, and the Y-axis points vertically downward. The light rod makes an angle $\theta$ with the vertical, and the coordinates of the point mass are $(x, y)$.

Find: Equations of motion using Lagrange’s equations using

a) one independent generalized coordinate, $\theta$

b) two dependent generalized coordinates, $(x, y)$

Solution:

a) Using $\theta$ as the generalized coordinate, the Lagrangian of the system can be written as

$$L = K - V = \frac{1}{2} mv^2 - (-mg) = \frac{1}{2} mL^2 \dot{\theta}^2 + mgL \cos(\theta)$$
Using Lagrange's equations in the form of Eq. (6), but with no non-conservative forces and no constraints, the equation of motion can be written as
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt} \left( mL^2 \ddot{\theta} \right) + mgL \sin(\theta) = mL^2 \ddot{\theta} + mgL \sin(\theta) = 0
\]
\[
\Rightarrow \ddot{\theta} + \frac{g}{L} \sin(\theta) = 0
\]

b) Using \((x, y)\) as the dependent generalized coordinates, the Lagrangian of the system can be written as
\[
L = K - V = \frac{1}{2} m v^2 - (-mg y) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mg y
\]
The two generalized coordinates can be related using the configuration constraint equation
\[
x^2 + y^2 = L^2
\]
Differentiating this equation with respect to time and writing the result in the form of Eq. (3) gives
\[
x\dot{x} + y\dot{y} = 0 \quad \text{or} \quad \left[ \begin{array}{c} x \\ y \\ \dot{x} \\ \dot{y} \end{array} \right] = 0 \quad \Rightarrow \quad a_{11} = x \quad a_{12} = y \quad a_{10} = 0
\]
Using Lagrange's equations in the form of Eq. (6) with no non-conservative forces and with one constraint gives
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \lambda a_{ik} \quad (k = 1, 2)
\]
Specifically,
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = \frac{d}{dt} \left( m \dot{x} \right) - 0 = m \ddot{x} = \lambda a_{11} = \lambda x \quad \Rightarrow \quad m \ddot{x} - \lambda x = 0
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = \frac{d}{dt} \left( m \dot{y} \right) - mg = m \ddot{y} - mg = \lambda a_{12} = \lambda y \quad \Rightarrow \quad m \ddot{y} - mg - \lambda y = 0
\]
Differentiating the constraint equation again gives
\[
x \dddot{x} + y \dddot{y} + \dot{x}^2 + \dot{y}^2 = 0
\]
These last three boxed equations are a complete set of three coupled, second-order, differential/algebraic equations in three unknowns \((x, y, \lambda)\).

Alternatively, the first two of the above equations can be used to eliminate the Lagrange multiplier \(\lambda\) from the equations. For example, if the first equation is multiplied by “\(y\)” and the second equation by “\(x\)” and the second equation is subtracted from the first, then the first two equations are reduced to a single equation. In this case, the equations of motion can be written as two coupled, second-order differential equations.
\[
\begin{align*}
y\ddot{x} - x\ddot{y} - gx &= 0 \\
x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0
\end{align*}
\]

**Note:** Clearly the use of a single coordinate is recommended for this simple system, but that will not always be the case. Both cases are shown here for easy comparison.

**Example 2: Equations of Motion of a Planar Slider-Crank Mechanism**

The equations of motion of a *slider-crank mechanism* can be formulated in various ways. The system shown to the right consists of a slider-crank mechanism which has had the slider constraint removed (or relaxed). The two links and the end-mass are all connected with simple revolute joints. The two links have masses \( m_i (i = 1, 2) \) and lengths \( \ell_i (i = 1, 2) \), and the end-mass has mass \( m_3 \). The end-mass is assumed to translate and not rotate.

One approach to finding the equations of motion of the slider-crank mechanism (shown in the lower figure) is to first find the equations of motion of the system in upper figure and then apply the necessary constraints to form the slider-crank mechanism shown in the lower figure.

Find: Using Lagrange’s equations with Lagrange multipliers,

a) equations of motion of the unconstrained system

b) equations of motion of the slider-crank mechanism

Assume that both systems shown move in a vertical plane and are driven by a torque \( T \) on the shaft at \( A \).

Solution:

a) **Kinetic Energy:**

The velocities of the mass center and the end of the second link can be written as

\[
\begin{align*}
\mathbf{v}_{G_2} &= \mathbf{v}_B + \mathbf{v}_{G_2/B} = \ell_1 \dot{\theta}_1 \hat{\mathbf{e}}_{\theta_1} + r_2 \dot{\theta}_2 \hat{\mathbf{e}}_{\theta_2} \\
\mathbf{v}_C &= \mathbf{v}_B + \mathbf{v}_{C/B} = \ell_1 \dot{\theta}_1 \hat{\mathbf{e}}_{\theta_1} + \ell_2 \dot{\theta}_2 \hat{\mathbf{e}}_{\theta_2}
\end{align*}
\]

Here, the unit vectors \( \hat{\mathbf{e}}_{\theta_i} \) \( (i = 1, 2) \) are perpendicular to the first and second links and directed in directions of increasing \( \theta_i \) \( (i = 1, 2) \). Using these results, the kinetic energy of the system can be written as
\[ K = K_1 + K_2 + K_3 = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{x}_{g2}^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_3 \dot{v}_c^2 \]
\[ = \frac{1}{2} \left( I_{G_1} + m_1 r_1^2 \right) \dot{\theta}_1^2 + \frac{1}{2} m_2 \left( \ell_1 \dot{\theta}_1 \varepsilon_{\theta_1} + r_2 \dot{\theta}_2 \varepsilon_{\theta_2} \right)^2 + \frac{1}{2} I_{G_2} \dot{\theta}_2^2 + \frac{1}{2} m_3 \left( \ell_1 \dot{\theta}_1 \varepsilon_{\theta_1} + \ell_2 \dot{\theta}_2 \varepsilon_{\theta_2} \right)^2 \]
\[ = \frac{1}{2} \left( I_{G_1} + m_1 r_1^2 \right) \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 r_2^2 \dot{\theta}_2^2 + m_2 \ell_2 r_2 \dot{\theta}_1 \dot{\theta}_2 c_{2-1} + \frac{1}{2} I_{G_2} \dot{\theta}_2^2 \]
\[ + \frac{1}{2} m_3 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_3 \ell_2^2 \dot{\theta}_2^2 + m_3 \ell_2 \dot{\theta}_1 \dot{\theta}_2 c_{2-1} \]
\[ \Rightarrow K = \frac{1}{2} \left( I_{G_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \dot{\theta}_1^2 + \frac{1}{2} \left( I_{G_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \dot{\theta}_2^2 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \dot{\theta}_2 c_{2-1} \]

Here, the symbol \( C_{2-1} \) is used to represent \( \cos(\theta_2 - \theta_1) \).

Potential Energy: (using a horizontal datum through point A)

The potential energy function associated with the weight forces can be written as

\[ V = V_1 + V_2 + V_3 = m_1 g r_1 S_1 + m_2 g \left( \ell_1 S_1 + r_2 S_2 \right) + m_3 g \left( \ell_2 S_1 + \ell_2 S_2 \right) \]
\[ = \left( m_1 r_1 + m_2 \ell_1 + m_3 \ell_1 \right) g S_1 + \left( m_2 r_2 + m_3 \ell_2 \right) g S_2 \]

In this expression, the symbols \( S_1 \) and \( S_2 \) are used to represent \( \sin(\theta_1) \) and \( \sin(\theta_2) \), respectively.

Lagrangian:

\[ L = K - V \]
\[ = \frac{1}{2} \left( I_{G_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \dot{\theta}_1^2 + \frac{1}{2} \left( I_{G_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \dot{\theta}_2^2 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \dot{\theta}_2 c_{2-1} \]
\[ - \left( m_1 r_1 + m_2 \ell_1 + m_3 \ell_1 \right) g S_1 - \left( m_2 r_2 + m_3 \ell_2 \right) g S_2 \]

Partial Derivatives of the Lagrangian:

\[ \frac{\partial L}{\partial \theta_1} = \left( I_{G_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \dot{\theta}_1 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_2 c_{2-1} \]
\[ \frac{\partial L}{\partial \theta_2} = \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} + \left( I_{G_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \dot{\theta}_2 \]
\[ \frac{\partial L}{\partial \theta_1} = \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} - \left( m_1 r_1 + m_2 \ell_1 + m_3 \ell_1 \right) g C_1 \]
\[ \frac{\partial L}{\partial \theta_2} = - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) g C_2 \]

Time Derivatives:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \theta_1} \right) = \left( I_{G_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \dot{\theta}_1 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_2 c_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ell_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) S_{2-1} \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) = \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_1 C_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) S_{2-1} + \left( I_{G_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \ddot{\theta}_2 \]

Lagrange’s Equations: (unconstrained system)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = \left( I_{G_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \ddot{\theta}_1 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \ddot{\theta}_2 C_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) S_{2-1} \\
- \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \ddot{\theta}_2 S_{2-1} + \left( m_1 r_1 + m_2 \ell_1 + m_3 \ell_1 \right) g C_1 \]

\[ \Rightarrow \left( I_{G_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \ddot{\theta}_1 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \ddot{\theta}_2 C_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) S_{2-1} \\
+ \left( m_1 r_1 + m_2 \ell_1 + m_3 \ell_1 \right) g C_1 = F_{\theta_1} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_1 C_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) S_{2-1} + \left( I_{G_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \ddot{\theta}_2 \\
+ \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \ddot{\theta}_2 S_{2-1} + \left( m_2 r_2 + m_3 \ell_2 \right) g C_2 \]

\[ \Rightarrow \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_1 C_{2-1} + \left( I_{G_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \ddot{\theta}_2 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \dot{\theta}_1 \ddot{\theta}_2 S_{2-1} \\
+ \left( m_2 r_2 + m_3 \ell_2 \right) g C_2 = F_{\theta_2} \]

Generalized Forces:

The contributions of the driving torque to the equations of motion can be written as

\[ F_{\theta_1} = T k \cdot \ddot{\theta}_1 / \dot{\theta}_1 = T k \cdot k = T \]
\[ F_{\theta_2} = T k \cdot \ddot{\theta}_1 / \dot{\theta}_2 = T k \cdot 0 = 0 \]

Summary:

The two equations of motion of the unconstrained system can be written as
b) To convert the unconstrained system into a simple slider-crank mechanism with zero offset (as shown), the mass \( m_3 \) must be constrained to move along a horizontal line passing through point A. The constraint equation associated with this restriction of motion is

\[
\ell_1 S_1 + \ell_2 S_2 = 0
\]

This constraint is converted to standard form by differentiating the expression with respect to time.

\[
\ell_1 C_1 \ddot{\theta}_1 + \ell_2 C_2 \ddot{\theta}_2 = 0
\]

or

\[
\begin{bmatrix} \ell_1 C_1 & \ell_2 C_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = 0
\]

Lagrange’s Equations: (for the constrained system)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = F \theta_i + \lambda a_{ii} \quad (i = 1, 2)
\]

Adding the Lagrange multiplier terms to the right side of the equations of motion of part (a) gives the equations of motion for the slider crank mechanism.

\[
\begin{align*}
\left( I_{\theta_1} + m_1 r_1^2 + m_2 \ell_1^2 + m_3 \ell_1^2 \right) \ddot{\theta}_1 &+ \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_2 C_{2-1} - \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_2 S_{2-1} \\
&+ \left( m_1 r_1 + m_2 \ell_1 + m_3 \ell_1 \right) g C_1 = T + \left( \ell_1 C_1 \right) \lambda
\end{align*}
\]

\[
\begin{align*}
\left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_1 C_{2-1} &+ \left( I_{\theta_2} + m_2 r_2^2 + m_3 \ell_2^2 \right) \ddot{\theta}_2 + \left( m_2 r_2 + m_3 \ell_2 \right) \ell_1 \ddot{\theta}_2 S_{2-1} \\
&+ \left( m_2 r_2 + m_3 \ell_2 \right) g C_2 = \left( \ell_2 C_2 \right) \lambda
\end{align*}
\]

These two equations are now supplemented with the twice differentiated constraint equation.

\[
\ell_1 C_1 \dddot{\theta}_1 + \ell_2 C_2 \dddot{\theta}_2 - \ell_1 S_1 \dot{\theta}_1^2 - \ell_2 S_2 \dot{\theta}_2^2 = 0
\]

These three equations represent a set of **three second-order, differential/algebraic equations** in three unknowns – \( \theta_1, \theta_2, \) and \( \lambda \).
Notes:

1. The use of a Lagrange multiplier in this problem allows construction of the equations of motion in terms of two variables instead of just one. This is a much easier alternative to developing the Lagrangian in terms of a single variable, say $\theta_1$, and then differentiating to find a single differential equation of motion (as would be required by Lagrange’s equations without a Lagrange multiplier). This clearly demonstrates the advantage of using Lagrange multipliers.

2. There are many ways of formulating the equations for complex systems using Lagrange multipliers. For example, consider developing the equations of motion of the slider crank mechanism using the following two systems.

First, develop the equations of motion of the individual systems. System 1 is a single degree-of-freedom system with one equation of motion for the angle $\theta_1$. System 2 is a two degree-of-freedom system with two equations of motion for the coordinate $x$ and the angle $\theta_2$. To form the slider-crank mechanism, the points labeled $B$ in each system must coincide. This gives rise to two constraint equations and two Lagrange multipliers.

$$\ell_1 C_1 - \ell_2 C_2 - x = 0 \quad \text{and} \quad \ell_1 S_1 - \ell_2 S_2 = 0$$

Modifying the original equations of motion using the constraint equations and their associated Lagrange multipliers gives three equations of motion in five unknowns – $\theta_1$, $\theta_2$, $x$, $\lambda_1$, and $\lambda_2$. Combining these equations with the differentiated constraint equations gives five equations in five unknowns.

3. Lagrange multipliers clearly make it easier to formulate the equations of motion. The cost of using them comes from having to solve a set of differential/algebraic equations instead of a smaller set of ordinary differential equations. Thus, the solution process is generally more detailed.
Example 3: Three-Dimensional Rotating Frame and Bar

The system shown has **two degrees of freedom** and consists of two bodies, the frame $F$ and the bar $B$. As $F$ rotates about the fixed vertical direction, $B$ rotates relative to the horizontal arm of $F$. The orientation of $F$ is given by the angle $\phi$ ($\dot{\phi} = \Omega$), and the orientation of $B$ is given by the angle $\theta$ ($\dot{\theta} = \omega$). The bar has mass $m$ and length $\ell$. The frame is assumed to be **light**. The motor torque $M_\phi(t)$ is applied to $F$ by the ground, and the motor torque $M_\theta(t)$ is applied to $B$ by $F$.

**Find:**

Using Kane’s equations, find the equations of motion of the system. Use the body-fixed angular velocity components of the bar as a set of dependent generalized speeds.

**Solution:**

The angular velocity of the bar can be written as

$$\omega_B = \dot{\theta} \mathbf{e}_2 + \dot{\phi} \mathbf{k} = \dot{\theta} \mathbf{e}_2 + \dot{\phi} (-S_\theta \mathbf{e}_1 + C_\theta \mathbf{e}_3) = -S_\theta \dot{\phi} \mathbf{e}_1 + \dot{\theta} \mathbf{e}_2 + C_\theta \dot{\phi} \mathbf{e}_3 \triangleq \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$$

From this expression, the angular velocity components $\omega_1$ and $\omega_3$ are **not independent**. The two can be related with the following constraint equation.

$$C_\theta \omega_1 + S_\theta \omega_3 = 0 \quad \text{or} \quad \begin{bmatrix} C_\theta & 0 & S_\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0$$

Given this set-up, Kane’s equations with a single Lagrange multiplier may be written as

$$\begin{bmatrix} m \ddot{R}_{G} - \frac{\partial^2 R_{V_G}}{\partial \omega_k} \end{bmatrix} + \left[ \begin{bmatrix} I_G \cdot \mathbf{R}_{B} \end{bmatrix} + \begin{bmatrix} R_{\omega_B} \times H_G \end{bmatrix} \right] \cdot \frac{\partial R_{\omega_B}}{\partial \omega_k} = F_{\omega_k} + \lambda a_{kk} \quad (k = 1, 2, 3)$$

**Kinematics:**

The velocity of the mass center of the bar can be written as

$$V_G = R_{\omega_B} \times z_{G/O} = \dot{\phi} k \times d n_2 = \dot{\phi} (-S_\theta \mathbf{e}_1 + C_\theta \mathbf{e}_3) \times d n_2 = (\omega_1 \mathbf{e}_1 + \omega_3 \mathbf{e}_3) \times d n_2$$

$$\Rightarrow \quad V_G = d \left( -\omega_3 \mathbf{e}_1 + \omega_1 \mathbf{e}_3 \right)$$
By direct differentiation, the acceleration of the mass center of the bar can be written as

\[ R \ddot{a}_G = d \left(-\dot{\omega}_3 \xi_1 + \dot{\omega}_1 \xi_3\right) + R \omega_B \times d \left(-\dot{\omega}_3 \xi_1 + \omega_1 \xi_3\right) = d \left(-\dot{\omega}_3 \xi_1 + \dot{\omega}_1 \xi_3\right) + d \left|\begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \\ \omega_1 & \omega_2 & \omega_3 \\ -\dot{\omega}_3 & 0 & \omega_1 \end{array}\right| \]

\[ \Rightarrow R \ddot{a}_G = d \left(-\dot{\omega}_3 + \omega_1 \omega_2 \right) \xi_1 - d \left(\omega_1^2 + \omega_3^2\right) \xi_2 + d \left(\dot{\omega}_1 + \omega_2 \omega_3\right) \xi_3 \]

Partial Angular Velocities and Partial Velocities:

The three partial angular velocities can be written as

\[ \frac{\partial R \omega_B}{\partial \omega_k} = \xi_k \quad (k = 1, 2, 3) \]

The three partial velocities can be written as

\[ \frac{\partial R \nu_G}{\partial \omega_1} = d \xi_3 \quad \frac{\partial R \nu_G}{\partial \omega_2} = 0 \quad \frac{\partial R \nu_G}{\partial \omega_3} = -d \xi_1 \]

Terms on the left side of Kane’s equations:

\[ m R \ddot{a}_G \cdot \frac{\partial R \omega_B}{\partial \omega_1} = m \left(d \left(-\dot{\omega}_3 + \omega_1 \omega_2 \right) \xi_1 - d \left(\omega_1^2 + \omega_3^2\right) \xi_2 + d \left(\dot{\omega}_1 + \omega_2 \omega_3\right) \xi_3\right) \cdot d \xi_3 = m d^2 \left(\dot{\omega}_3 - \omega_1 \omega_2\right) \]

\[ m R \ddot{a}_G \cdot \frac{\partial R \nu_G}{\partial \omega_2} = 0 \]

\[ m R \ddot{a}_G \cdot \frac{\partial R \nu_G}{\partial \omega_3} = m \left(d \left(-\dot{\omega}_3 + \omega_1 \omega_2 \right) \xi_1 - d \left(\omega_1^2 + \omega_3^2\right) \xi_2 + d \left(\dot{\omega}_1 + \omega_2 \omega_3\right) \xi_3\right) \cdot (-d \xi_1) = m d^2 \left(\dot{\omega}_1 - \omega_3 \omega_2\right) \]

\[ I_G \cdot R \ddot{a}_B \Rightarrow \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} \Rightarrow I_G \cdot R \ddot{a}_B = \begin{bmatrix} I_1 \dot{\omega}_1 \xi_1 + I_2 \dot{\omega}_2 \xi_2 + I_3 \dot{\omega}_3 \xi_3 \end{bmatrix} \]

\[ \left(I_G \cdot R \ddot{a}_B\right) \cdot \frac{\partial R \omega_B}{\partial \omega_k} = I_k \dot{\omega}_k \quad (k = 1, 2, 3) \]

\[ H_G = I_G \cdot R \omega_B \Rightarrow \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Rightarrow H_G = \begin{bmatrix} I_1 \dot{\omega}_1 \xi_1 + I_2 \dot{\omega}_2 \xi_2 + I_3 \dot{\omega}_3 \xi_3 \end{bmatrix} \]
\[ R \mathbf{Q}_B \times \mathbf{H}_G = \begin{vmatrix} \varepsilon_1 & n_2 & \varepsilon_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1 \omega_1 & I_2 \omega_2 & I_3 \omega_3 \end{vmatrix} = (I_3 - I_2) \omega_2 \omega_3 \varepsilon_1 + (I_1 - I_3) \omega_1 \omega_3 n_2 + (I_2 - I_1) \omega_1 \omega_2 \varepsilon_3 \]

\[ \left( R \mathbf{Q}_B \times \mathbf{H}_G \right) \cdot \frac{\partial R \mathbf{Q}_B}{\partial \omega_1} = (I_3 - I_2) \omega_2 \omega_3 \]

\[ \left( R \mathbf{Q}_B \times \mathbf{H}_G \right) \cdot \frac{\partial R \mathbf{Q}_B}{\partial \omega_2} = (I_1 - I_3) \omega_1 \omega_3 \]

\[ \left( R \mathbf{Q}_B \times \mathbf{H}_G \right) \cdot \frac{\partial R \mathbf{Q}_B}{\partial \omega_3} = (I_2 - I_1) \omega_1 \omega_2 \]

### Generalized Forces:

The three generalized forces associated with the driving torques can be written as

\[
F_{\omega_1} = \left[ M_\phi \left( \varepsilon_1 + M_\theta n_2 \right) \right] + \left[ M_\theta \left( -S_\theta \varepsilon_1 + C_\theta \varepsilon_3 \right) \right] = -S_\theta M_\phi
\]

\[
F_{\omega_2} = \left[ M_\phi \left( \varepsilon_2 + M_\theta n_2 \right) \right] + \left[ M_\theta \left( -S_\theta \varepsilon_2 + C_\theta \varepsilon_3 \right) \right] = M_\theta
\]

\[
F_{\omega_3} = \left[ M_\phi \left( \varepsilon_3 + M_\theta n_2 \right) \right] + \left[ M_\theta \left( -S_\theta \varepsilon_3 + C_\theta \varepsilon_3 \right) \right] = C_\theta M_\phi
\]

Note here that the torque \( M_\phi \) is transferred directly to the bar \( B \) because the frame \( F \) is light.

### Kane’s Equations of Motion:

Substituting the intermediate results into Kane’s equations gives

\[
k = 1: \quad m d^2 \left( \dot{\omega}_1 + \omega_2 \omega_3 \right) + I_3 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = -S_\theta M_\phi + C_\theta \lambda
\]

\[
\Rightarrow \left( I_1 + m d^2 \right) \dot{\omega}_1 + (I_3 + m d^2 - I_2) \omega_2 \omega_3 = -S_\theta M_\phi + C_\theta \lambda
\]

\[
k = 2: \quad I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = M_\theta
\]

\[
k = 3: \quad m d^2 \left( \dot{\omega}_3 - \omega_1 \omega_2 \right) + I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = C_\theta M_\phi + S_\theta \lambda
\]

\[
\Rightarrow \left( I_3 + m d^2 \right) \dot{\omega}_3 + (I_2 - I_1 - m d^2) \omega_1 \omega_2 = C_\theta M_\phi + S_\theta \lambda
\]
Note:
If the first of these equations is multiplied by \(-S_{\theta}\) and the third by \(C_{\theta}\) and the resulting equations added together, the Lagrange multiplier can be eliminated from the equations. Further, if the angular velocity component \(\omega_3\) is written in terms of \(\omega_1\), then the two resulting equations (along with a set of kinematic equations relating the angular velocity components \(\omega_1\) and \(\omega_2\) to \(\dot{\phi}\) and \(\dot{\theta}\)) can be solved without regard to the Lagrange multiplier. Following this process and treating the bar as slender, the above equations can be shown to be identical to those found in previous Units using Lagrange’s equations and d’Alembert’s principle.

Example 4: Double Pendulum

The system shown is a three-dimensional double pendulum or arm. The first link is connected to ground and the second link is connected to the first with ball and socket joints at \(O\) and \(A\). The orientation of each link is defined relative to the ground using a 3-1-3 body-fixed rotation sequence. The lengths of the links are \(\ell_1\) and \(\ell_2\). The links are assumed to be slender bars with mass centers at their midpoints. Assume the \(\mathbb{N}_2\) direction is vertical.

Reference frames:
\[ R : N_1, N_2, N_3 \quad \text{(fixed frame)} \]
\[ L_i : n_{1i}^i, n_{2i}^i, n_{3i}^i \quad (i = 1, 2) \quad \text{(fixed in the two links)} \]

Find:
Using Kane’s equations with Lagrange multipliers, find the equations of motion describing the free motion of the double pendulum under the action of gravity. Use the nine dependent generalized speeds
\[ \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_{10}, u_{11}, u_{12}\} = \{\omega_{11}, \omega_{12}, \omega_{13}, \omega_{21}, \omega_{22}, \omega_{23}, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}\} \]

The first six generalized speeds are the body-fixed angular velocity components of the two links, and the last six generalized speeds are the inertial components of the velocities of the mass centers of links.

\[
\begin{align*}
R_{L_1} &= \omega_{11} \ n_{11}^1 + \omega_{12} \ n_{12}^1 + \omega_{13} \ n_{13}^1 \\
R_{L_2} &= \omega_{21} \ n_{21}^2 + \omega_{22} \ n_{22}^2 + \omega_{23} \ n_{23}^2 \\
R_{G_1} &= v_{11} \ n_{11}^1 + v_{12} \ n_{21}^1 + v_{13} \ n_{31}^1 \\
R_{G_2} &= v_{21} \ n_{11}^2 + v_{22} \ n_{21}^2 + v_{23} \ n_{31}^2 
\end{align*}
\]
Solution:

Transformation Matrices and Angular Velocities: (previous results)

The **transformation matrices** and the **body-fixed components** of the **angular velocities** of the links $L_i$ ($i = 1, 2$) are as given in Unit 5 of Volume I.

\[
[R_i] = \begin{bmatrix}
C_{i1}C_{i3} - S_{i1}C_{i2}S_{i3} & S_{i1}C_{i3} + C_{i1}C_{i2}S_{i3} & S_{i2}S_{i3} \\
-C_{i1}S_{i3} - S_{i1}C_{i2}C_{i3} & -S_{i1}S_{i3} + C_{i1}C_{i2}C_{i3} & S_{i2}C_{i3} \\
S_{i1}S_{i2} & -C_{i1}S_{i2} & C_{i2}
\end{bmatrix}
\]

\[
\omega_{i1} = \dot{\theta}_{i1}S_{i2}S_{i3} + \dot{\theta}_{i2}C_{i3} \\
\omega_{i2} = \dot{\theta}_{i1}S_{i2}C_{i3} - \dot{\theta}_{i2}S_{i3} \\
\omega_{i3} = \dot{\theta}_{i3} + \dot{\theta}_{i1}C_{i2}
\]

Angular Momentum: (previous results)

The **inertia matrices** of the links about their mass centers can be written as

\[
[I_{Gi}]_{Li} = \begin{bmatrix}
I_{11} & 0 & 0 \\
0 & I_{22} & 0 \\
0 & 0 & I_{33}
\end{bmatrix} \quad (i = 1, 2)
\]

Here, because the links are assumed to be slender bars, inertias about the $n_2^i$ and $n_3^i$ directions are identical and the inertias about the $n_1^i$ directions are small. Using these inertia matrices, the **angular momenta** of the links about their mass centers can be written as

\[
H_{Gi} = [I_{Gi}]_{Li} \cdot R \omega_{Li} \Rightarrow \begin{bmatrix}
I_{11} & 0 & 0 \\
0 & I_{22} & 0 \\
0 & 0 & I_{33}
\end{bmatrix} \begin{bmatrix}
\omega_{i1} \\
\omega_{i2} \\
\omega_{i3}
\end{bmatrix} \Rightarrow H_{Gi} = I_{11}^i \omega_{i1}^i n_1^i + I_{22}^i \omega_{i2}^i n_2^i + I_{33}^i \omega_{i3}^i n_3^i \quad (i = 1, 2)
\]

Finally, the cross product of the angular velocity with the angular momentum for each of the links can be written as

\[
[R \omega_{Li}] \times H_{Gi} = \begin{bmatrix}
\dot{n}_1^i & \dot{n}_2^i & \dot{n}_3^i \\
\dot{\omega}_{i1} & \dot{\omega}_{i2} & \dot{\omega}_{i3} \\
I_{11}^i \omega_{i1}^i & I_{22}^i \omega_{i2}^i & I_{33}^i \omega_{i3}^i
\end{bmatrix} = (I_{33}^i - I_{22}^i) \omega_{i2}^i \omega_{i3}^i n_1^i + (I_{11}^i - I_{33}^i) \omega_{i1}^i \omega_{i3}^i n_2^i + (I_{22}^i - I_{11}^i) \omega_{i1}^i \omega_{i2}^i n_3^i \quad (i = 1, 2)
\]

Constraint Equations:

The **generalized speeds** associated with translational and rotational motion can be related by first finding the **fixed-frame components** of the **position vectors** of the links. Then, these components can be **differentiated** to relate the generalized speeds.
The fixed-frame position vector components of mass center $G_1$ can be written in matrix form as

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ -\frac{1}{2} \ell_1 \\ 0 \end{bmatrix} = -\frac{1}{2} \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using results from Addendum 2 of Unit 4 of this volume, this result can be differentiated.

$$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = -\frac{1}{2} \ell_1 \begin{bmatrix} \ddot{R}_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} \ddot{\omega} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ -\omega_{13} \\ \omega_{12} \end{bmatrix} \begin{bmatrix} 0 \\ -\omega_{12} \\ \omega_{11} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} + \frac{1}{2} \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} -\omega_{13} \\ 0 \\ \omega_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This second term on the left side of this expression can be expanded to give

$$\frac{1}{2} \ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} -\omega_{13} \\ 0 \\ \omega_{11} \end{bmatrix} = \frac{1}{2} \ell_1 \begin{bmatrix} R_1^T \omega_{13} + R_1^T \omega_{11} \\ -R_1^T \omega_{13} + R_1^T \omega_{11} \\ -R_1^T \omega_{13} + R_1^T \omega_{11} \end{bmatrix} = \frac{1}{2} \ell_1 \begin{bmatrix} R_1^T \omega_{13} \\ R_1^T \omega_{13} \\ -R_1^T \omega_{13} \end{bmatrix}$$

Substituting the expanded results into the previous equation gives

$$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} + \frac{1}{2} \ell_1 \begin{bmatrix} R_1^T \omega_{13} \\ R_1^T \omega_{13} \\ -R_1^T \omega_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The fixed-frame position vector components of mass center $G_2$ can be written in matrix form as follows.

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ -\ell_1 \\ 0 \end{bmatrix} + \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} 0 \\ -\frac{1}{2} \ell_2 \\ 0 \end{bmatrix} = -\ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} -\frac{1}{2} \ell_2 \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Again, using results from Addendum 2 of Unit 4 of this volume, this result can be differentiated as follows.

$$\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = -\ell_1 \begin{bmatrix} \ddot{R}_1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} -\frac{1}{2} \ell_2 \begin{bmatrix} \ddot{R}_2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\ell_1 \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} \ddot{\omega} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} -\frac{1}{2} \ell_2 \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} \ddot{\omega} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
\[
\begin{bmatrix}
\ell_1 [R_1]^T \\
\ell_1 [R_2]^T
\end{bmatrix}
\begin{bmatrix}
0 & -\omega_{13} & \omega_{12} \\
\omega_{13} & 0 & -\omega_{11} \\
-\omega_{12} & \omega_{11} & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
= \ell_1 [-\frac{1}{2} R_1 + \frac{1}{2} R_2]
\begin{bmatrix}
0 & -\omega_{23} & \omega_{22} \\
-\omega_{23} & 0 & -\omega_{21} \\
-\omega_{22} & \omega_{21} & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

\[
\Rightarrow
\begin{bmatrix}
v_{21} \\
v_{22} \\
v_{23}
\end{bmatrix} + \ell_1 [R_1]^T \begin{bmatrix}
-\omega_{13} \\
0 \\
\omega_{11}
\end{bmatrix} + \frac{1}{2} \ell_2 [R_2]^T \begin{bmatrix}
-\omega_{23} \\
0 \\
\omega_{21}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

The second and third terms on the left side of this equation can be expanded to give

\[
\ell_1 [R_1]^T \begin{bmatrix}
-\omega_{13} \\
0 \\
\omega_{11}
\end{bmatrix} + \frac{1}{2} \ell_2 [R_2]^T \begin{bmatrix}
-\omega_{23} \\
0 \\
\omega_{21}
\end{bmatrix} = \ell_1 \begin{bmatrix}
-R_{11}^1 \omega_{13} + R_{31}^1 \omega_{11} \\
-R_{12}^1 \omega_{13} + R_{32}^1 \omega_{11} \\
-R_{13}^1 \omega_{13} + R_{33}^1 \omega_{11}
\end{bmatrix} + \frac{1}{2} \ell_2 \begin{bmatrix}
-R_{11}^2 \omega_{23} + R_{31}^2 \omega_{21} \\
-R_{12}^2 \omega_{23} + R_{32}^2 \omega_{21} \\
-R_{13}^2 \omega_{23} + R_{33}^2 \omega_{21}
\end{bmatrix}
\]

Substituting the expanded results into the previous result gives

\[
\begin{bmatrix}
v_{21} \\
v_{22} \\
v_{23}
\end{bmatrix} + \ell_1 \begin{bmatrix}
R_{31}^1 & 0 & -R_{11}^1 \\
R_{32}^1 & 0 & -R_{12}^1 \\
R_{33}^1 & 0 & -R_{13}^1
\end{bmatrix} \begin{bmatrix}
\omega_{11} \\
\omega_{12} \\
\omega_{13}
\end{bmatrix} + \frac{1}{2} \ell_2 \begin{bmatrix}
R_{31}^2 & 0 & -R_{11}^2 \\
R_{32}^2 & 0 & -R_{12}^2 \\
R_{33}^2 & 0 & -R_{13}^2
\end{bmatrix} \begin{bmatrix}
\omega_{21} \\
\omega_{22} \\
\omega_{23}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Combining the constraint equations and rearranging into the standard form for constraint equations gives the following.

\[
[A]_{6 \times 12} \{u\}_{12 \times 1} = \{0\}_{6 \times 1}
\]

The coefficient matrix \([A]\) is shown here partitioned into a set of 3×3 matrices.
The vector of generalized speeds is shown partitioned into a set of 3×1 vectors.

\[
\{u\}_{12×1} = \begin{bmatrix}
\omega_{11} & \omega_{12} & \omega_{13} \\
\omega_{21} & \omega_{22} & \omega_{23} \\
v_{11} & v_{12} & v_{13} \\
v_{21} & v_{22} & v_{23}
\end{bmatrix}^T
\]

Kane’s Equation of Motion:

\[
\sum_{i=1}^{2} \left( m_i \dot{a}_{G_i} \cdot \frac{\partial R\dot{y}_{G_i}}{\partial u_k} \right) + \sum_{i=1}^{2} \left( \left( I_{zG_i} \cdot R\ddot{\alpha}_{L_i} \right) + \left( R\omega_{\alpha_{L_i}} \times H_{G_i} \right) \right) \cdot \frac{\partial R\dot{\omega}_{L_i}}{\partial u_k} = F_{uk} + \sum_{j=1}^{6} \lambda_j a_{jk} \quad (k = 1, \ldots, 9)
\]

Terms associated with translational motion:

\[
\begin{align*}
m_1 \frac{R\dot{a}_{G_1}}{\partial u_k} \cdot \frac{\partial R\dot{y}_{G_1}}{\partial u_k} &= 0 \quad (k = 1, \ldots, 6) \text{ and } (k = 10, 11, 12) \\
m_1 \frac{R\dot{a}_{G_1}}{\partial u_7} \cdot \frac{\partial R\dot{y}_{G_1}}{\partial u_7} &= m_1 \dot{v}_{11} \\
m_1 \frac{R\dot{a}_{G_1}}{\partial u_8} \cdot \frac{\partial R\dot{y}_{G_1}}{\partial u_8} &= m_1 \dot{v}_{12} \\
m_1 \frac{R\dot{a}_{G_1}}{\partial u_9} \cdot \frac{\partial R\dot{y}_{G_1}}{\partial u_9} &= m_1 \dot{v}_{13} \\
m_2 \frac{R\dot{a}_{G_2}}{\partial u_k} \cdot \frac{\partial R\dot{y}_{G_2}}{\partial u_k} &= 0 \quad (k = 1, \ldots, 9) \\
m_2 \frac{R\dot{a}_{G_2}}{\partial u_{10}} \cdot \frac{\partial R\dot{y}_{G_2}}{\partial u_{10}} &= m_2 \dot{v}_{21} \\
m_2 \frac{R\dot{a}_{G_2}}{\partial u_{11}} \cdot \frac{\partial R\dot{y}_{G_2}}{\partial u_{11}} &= m_2 \dot{v}_{22} \\
m_2 \frac{R\dot{a}_{G_2}}{\partial u_{12}} \cdot \frac{\partial R\dot{y}_{G_2}}{\partial u_{12}} &= m_2 \dot{v}_{23}
\end{align*}
\]

Terms associated with rotational motion:

\[
\begin{align*}
\left( I_{zG_1} \cdot R\ddot{\alpha}_{L_1} \right) + \left( R\omega_{\alpha_{L_1}} \times H_{G_1} \right) \cdot \frac{\partial R\dot{\omega}_{L_1}}{\partial u_1} &= I_{11}^1 \dot{\omega}_{11} + \left( I_{33}^1 - I_{22}^1 \right) \omega_{12} \omega_{13} \\
\left( I_{zG_1} \cdot R\ddot{\alpha}_{L_1} \right) + \left( R\omega_{\alpha_{L_1}} \times H_{G_1} \right) \cdot \frac{\partial R\dot{\omega}_{L_1}}{\partial u_2} &= I_{12}^1 \dot{\omega}_{12} + \left( I_{11}^1 - I_{33}^1 \right) \omega_{11} \omega_{13} \\
\left( I_{zG_1} \cdot R\ddot{\alpha}_{L_1} \right) + \left( R\omega_{\alpha_{L_1}} \times H_{G_1} \right) \cdot \frac{\partial R\dot{\omega}_{L_1}}{\partial u_3} &= I_{13}^1 \dot{\omega}_{13} + \left( I_{22}^1 - I_{11}^1 \right) \omega_{12} \omega_{11} \\
\left( I_{zG_1} \cdot R\ddot{\alpha}_{L_1} \right) + \left( R\omega_{\alpha_{L_1}} \times H_{G_1} \right) \cdot \frac{\partial R\dot{\omega}_{L_1}}{\partial u_k} &= 0 \quad (k = 4, \ldots, 12) \\
\left( I_{zG_2} \cdot R\ddot{\alpha}_{L_2} \right) + \left( R\omega_{\alpha_{L_2}} \times H_{G_2} \right) \cdot \frac{\partial R\dot{\omega}_{L_2}}{\partial u_k} &= 0 \quad (k = 1, 2, 3) \text{ and } (k = 7, \ldots, 12)
\end{align*}
\]
Generalized Forces:

The only two active forces are the weight forces that act at the mass centers of the two links. The generalized forces associated with these two forces can be written as

$$ F_{u_k} = \left[ W_1 \cdot \frac{\partial \gamma_{G_1}}{\partial_{u_k}} \right] + \left[ W_2 \cdot \frac{\partial \gamma_{G_2}}{\partial_{u_k}} \right] (k = 1, \ldots, 12) $$

Specifically,

$$ F_{u_k} = \left[ -m_1 g N_2 \right] \cdot \frac{\partial \gamma_{G_1}}{\partial_{u_k}} = 0 $$

$$ F_{u_8} = \left[ -m_1 g N_2 \right] \cdot \frac{\partial \gamma_{G_1}}{\partial_{u_8}} = -m_1 g $$

$$ F_{u_0} = \left[ -m_2 g N_2 \right] \cdot \frac{\partial \gamma_{G_1}}{\partial_{u_7}} = 0 $$

$$ F_{u_{11}} = \left[ -m_2 g N_2 \right] \cdot \frac{\partial \gamma_{G_2}}{\partial_{u_{11}}} = -m_2 g $$

$$ F_{u_{12}} = \left[ -m_2 g N_2 \right] \cdot \frac{\partial \gamma_{G_2}}{\partial_{u_{12}}} = 0 $$
Substituting into Kane’s equations of motion:

Equation for \( u_1 = \omega_{11} \):
\[
I_{11}^1 \dot{\omega}_{11} + (I_{13}^1 - I_{22}^1) \omega_{12} \omega_{13} = \frac{1}{2} \ell_1 (R_{31}^1) \dot{\lambda}_1 + \frac{1}{2} \ell_1 (R_{32}^1) \dot{\lambda}_2 + \frac{1}{2} \ell_1 (R_{33}^1) \dot{\lambda}_3 + \ell_1 (R_{31}^1) \dot{\lambda}_4 + \ell_1 (R_{32}^1) \dot{\lambda}_5 + \ell_1 (R_{33}^1) \dot{\lambda}_6
\]  
(10)

Equation for \( u_2 = \omega_{12} \):
\[
I_{22}^1 \dot{\omega}_{12} + (I_{11}^1 - I_{13}^1) \omega_{11} \omega_{13} = 0
\]  
(11)

Equation for \( u_3 = \omega_{13} \):
\[
I_{33}^1 \dot{\omega}_{13} + (I_{22}^1 - I_{11}^1) \omega_{11} \omega_{12} = -\frac{1}{2} \ell_1 (R_{11}^1) \dot{\lambda}_1 - \frac{1}{2} \ell_1 (R_{12}^1) \dot{\lambda}_2 - \frac{1}{2} \ell_1 (R_{13}^1) \dot{\lambda}_3 - \ell_1 (R_{11}^1) \dot{\lambda}_4 - \ell_1 (R_{12}^1) \dot{\lambda}_5 - \ell_1 (R_{13}^1) \dot{\lambda}_6
\]  
(12)

Equation for \( u_4 = \omega_{21} \):
\[
I_{11}^2 \dot{\omega}_{21} + (I_{33}^1 - I_{22}^1) \omega_{22} \omega_{23} = \frac{1}{2} \ell_2 (R_{31}^2) \dot{\lambda}_4 + \frac{1}{2} \ell_2 (R_{32}^2) \dot{\lambda}_5 + \frac{1}{2} \ell_2 (R_{33}^2) \dot{\lambda}_6
\]  
(13)

Equation for \( u_5 = \omega_{22} \):
\[
I_{22}^2 \dot{\omega}_{22} + (I_{11}^2 - I_{33}^1) \omega_{21} \omega_{23} = 0
\]  
(14)

Equation for \( u_6 = \omega_{23} \):
\[
I_{33}^2 \dot{\omega}_{23} + (I_{22}^2 - I_{11}^2) \omega_{21} \omega_{22} = -\frac{1}{2} \ell_2 (R_{11}^2) \dot{\lambda}_4 - \frac{1}{2} \ell_2 (R_{12}^2) \dot{\lambda}_5 - \frac{1}{2} \ell_2 (R_{13}^2) \dot{\lambda}_6
\]  
(15)

Equation for \( u_7 = \dot{x}_1 \):
\[
m_1 \ddot{x}_{11} = \dot{\lambda}_1
\]  
(16)

Equation for \( u_8 = \dot{y}_1 \):
\[
m_1 \ddot{y}_{12} = -m_1 g + \dot{\lambda}_2
\]  
(17)

Equation for \( u_9 = \dot{z}_1 \):
\[
m_1 \ddot{z}_{13} = \dot{\lambda}_3
\]  
(18)

Equation for \( u_{10} = \dot{x}_2 \):
\[
m_2 \ddot{x}_{21} = \dot{\lambda}_4
\]  
(19)

Equation for \( u_{11} = \dot{y}_2 \):
\[
m_2 \ddot{y}_{22} = -m_2 g + \dot{\lambda}_5
\]  
(20)

Equation for \( u_{12} = \dot{z}_2 \):
\[
m_2 \ddot{z}_{23} = \dot{\lambda}_6
\]  
(21)

Differentiated Constraint Equations:
As determined above, the constraint equations relating the inertial mass-center velocities components and the body-fixed angular velocity components are
\[
\begin{bmatrix}
v_{11} \\
v_{12} \\
v_{13}
\end{bmatrix} = \frac{1}{2} \ell_1 [R_1] \dot{\omega}_{13} \\
\begin{bmatrix}
0 \\
-\omega_{11}
\end{bmatrix}
\]
\[
\begin{bmatrix}
v_{21} \\
v_{22} \\
v_{23}
\end{bmatrix} = \ell_1 [R_1] \dot{\omega}_{13} \\
\begin{bmatrix}0 \\
-\omega_{11}
\end{bmatrix} + \frac{1}{2} \ell_2 [R_2] \dot{\omega}_{23} \\
\begin{bmatrix}0 \\
-\omega_{21}
\end{bmatrix}
\]
These equations can be differentiated again to form a set of six, second-order differential equations.

Differentiating the first set of equations gives

\[
\begin{align*}
\dot{\mathbf{v}}_{11} &= \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \dot{\omega}_{13} 0 \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ 0 -\dot{\omega}_{11} \right\} = \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ \ddot{\phi}_{13} \right\} = 0 \\
\dot{\mathbf{v}}_{12} &= \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \dot{\omega}_{13} 0 \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ 0 -\dot{\omega}_{11} \right\} = \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ \ddot{\phi}_{13} \right\} = 0 \\
\dot{\mathbf{v}}_{13} &= \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \dot{\omega}_{13} 0 \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ 0 -\dot{\omega}_{11} \right\} = \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ \ddot{\phi}_{13} \right\} = 0
\end{align*}
\]

\[
\Rightarrow \begin{align*}
\dot{\mathbf{v}}_{11} &= -\frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} = 0 \\
\dot{\mathbf{v}}_{12} &= -\frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} = 0 \\
\dot{\mathbf{v}}_{13} &= -\frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} = 0
\end{align*}
\]

Differentiating the second set of equations gives

\[
\begin{align*}
\dot{\mathbf{v}}_{21} &= \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_2 \left[ \mathbf{R}_2^T \right]^T \left\{ 0 -\phi_{23} \right\} + \ell_1 \left[ \mathbf{\dot{R}}_1^T \right]^T \left\{ 0 -\dot{\phi}_{21} \right\} = \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_2 \left[ \mathbf{R}_2^T \right]^T \left\{ \ddot{\phi}_{23} \right\} = 0 \\
\dot{\mathbf{v}}_{22} &= \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_2 \left[ \mathbf{R}_2^T \right]^T \left\{ 0 -\phi_{23} \right\} = \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_2 \left[ \mathbf{R}_2^T \right]^T \left\{ \ddot{\phi}_{23} \right\} = 0 \\
\dot{\mathbf{v}}_{23} &= \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_2 \left[ \mathbf{R}_2^T \right]^T \left\{ 0 -\phi_{23} \right\} = \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} + \frac{1}{2} \ell_2 \left[ \mathbf{R}_2^T \right]^T \left\{ \ddot{\phi}_{23} \right\} = 0
\end{align*}
\]

\[
\Rightarrow \begin{align*}
\dot{\mathbf{v}}_{21} &= \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} = 0 \\
\dot{\mathbf{v}}_{22} &= \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} = 0 \\
\dot{\mathbf{v}}_{23} &= \ell_1 \left[ \mathbf{R}_1^T \right]^T \left\{ \phi_{13} \right\} = 0
\end{align*}
\]
Eqs. (10) through (23) form a set of **eighteen, first-order, differential/algebraic** equations in as many unknowns \(- \omega_1, \omega_1, \omega_1, \omega_21, \omega_22, \omega_23, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6.\)

It can be shown that Eqs. (10) through (23) are equivalent to those found in Example 6 of Unit 5. In that example, Kane’s equations were used to generate the equations of motion for a set of **six independent generalized speeds** – the body-fixed angular velocity components. To show the equations are equivalent, first solve Eqs. (16) through (21) for the Lagrange multipliers in terms of the six velocity components. Then substitute those results into the right sides of Eqs. (10), (12), (13), and (15). Combining those terms into a matrix form, the differentiated constraint equations (22) and (23) can be used to eliminate the velocity components in favor of the angular velocity components. The resulting equations are identical to those found in Unit 5.

**Exercises:**

6.1 In exercises in Units 4 and 5 of this volume, equations of motion were formulated for this system using a single generalized coordinate \(\theta.\)

a) Using Lagrange’s equations or d’Alembert’s principle, formulate the equations of motion of the system using the set of **dependent** generalized coordinates \((x, y, \theta).\) There are **five** differential equations in all. The equations will contain five variables: \((x, y, \theta)\) and two Lagrange multipliers \((\lambda_1, \lambda_2).\) b) By eliminating \(\lambda_1, \lambda_2, x,\) and \(y\) from the equations of motion, show that the five equations found in part (a) are equivalent to the single equation found in Exercises 4.3 and 5.1.

Answers:

- **Three differential/algebraic equations of motion**
  
  \[
  \frac{1}{4}m \ddot{x} + c \dot{x} + k x = \lambda_1 \\
  (m_p + \frac{1}{4}m) \ddot{y} + (m_p + \frac{1}{2}m) g = \lambda_2 - F(t) \\
  \left(\frac{1}{12} m t^2\right) \ddot{\theta} = (-\ell C_\theta) \lambda_1 + (\ell S_\theta) \lambda_2
  \]

- **Two differentiated constraint equations**
  
  \[
  \ddot{x} - (\ell C_\theta) \dot{\theta} + (\ell S_\theta) \dot{\theta}^2 = 0 \\
  \ddot{y} + (\ell S_\theta) \dot{\theta} + (\ell C_\theta) \dot{\theta}^2 = 0
  \]

6.2 In exercises in Units 4 and 5 of this volume, equations of motion were formulated for this system using two generalized coordinates \(\theta\) and \(x.\) a) Using Lagrange’s equations or d’Alembert’s principle, formulate the equations of motion of the system using the set of **dependent** generalized coordinates \((x, x_G, y_G, \theta).\) Here, \((x_G, y_G)\) represent the \(X\) and \(Y\) coordinates of \(G\) relative to \(A.\) There are **six** differential equations in all.
The equations will contain six variables: \((x, x_G, y_G, \theta)\) and two Lagrange multipliers \((\lambda_1, \lambda_2)\). b) By eliminating \(x_G, y_G, \lambda_1, \) and \(\lambda_2\) from the equations of motion, show the six equations found in part (a) are equivalent to the two equations of motion found in Exercises 4.4 and 5.2.

Answers:

Four differential/algebraic equations of motion

\[
\begin{align*}
(m_1 + m_2)\ddot{x} + m_2\ddot{x}_G + cx + kx &= F(t) \\
m_2\ddot{x} + m_2\ddot{x}_G &= \lambda_1 \\
m_2\ddot{y} - m_2g &= \lambda_2 \\
\left(\frac{1}{12}m_2\ell^2\right)\ddot{\theta} &= \left(-\frac{1}{2}C_\theta\right)\lambda_1 + \left(\frac{1}{2}S_\theta\right)\lambda_2
\end{align*}
\]

Two differentiated constraint equations

\[
\begin{align*}
\ddot{x}_G - \left(\frac{1}{2}C_\theta\right)\ddot{\theta} + \left(\frac{1}{2}S_\theta\right)\dot{\theta}^2 &= 0 \\
\ddot{y}_G + \left(\frac{1}{2}C_\theta\right)\ddot{\theta} + \left(\frac{1}{2}S_\theta\right)\dot{\theta}^2 &= 0
\end{align*}
\]

6.3 Using Lagrange’s equations or d’Alembert’s principle, formulate the equations of motion of a slider-crank mechanism based on the two systems shown. Use \((\theta_1, \theta_2, x)\) as three dependent generalized coordinates. There is a total of \textbf{five} differential equations for the five variables: three coordinates \((\theta_1, \theta_2, x)\) and two Lagrange multipliers \((\lambda_1, \lambda_2)\).

Answers:

Equations of motion

\[
\begin{align*}
\left(I_{G_1} + m_1r_1^2\right)\ddot{\theta}_1 + m_1gr_1C_{\theta_1} &= T + (\ell_1C_{\theta_1})\lambda_1 - (\ell_1S_{\theta_1})\lambda_2 \\
\left(I_{G_2} + m_2r_2^2\right)\ddot{\theta}_2 - m_2r_2S_{\theta_2}\ddot{x} + m_2gr_2C_{\theta_2} &= -(\ell_2C_{\theta_2})\lambda_1 + (\ell_2S_{\theta_2})\lambda_2 \\
\left(m_2 + m_3\right)\ddot{x} - m_2r_2S_{\theta_2}\ddot{\theta}_2 - m_2r_2C_{\theta_2}\dot{\theta}_2^2 &= -\lambda_2
\end{align*}
\]

Constraint equations

\[
\begin{align*}
\ell_1C_{\theta_1}\ddot{\theta}_1 - \ell_2C_{\theta_2}\ddot{\theta}_2 - \ell_1S_{\theta_1}\dot{\theta}_1^2 + \ell_2S_{\theta_2}\dot{\theta}_2^2 &= 0 \\
-\ell_1S_{\theta_1}\ddot{\theta}_1 + \ell_2S_{\theta_2}\ddot{\theta}_2 - \ddot{x} - \ell_1C_{\theta_1}\dot{\theta}_1^2 + \ell_2C_{\theta_2}\dot{\theta}_2^2 &= 0
\end{align*}
\]
6.4 Find the differential equations of motion of the two degree-of-freedom system shown using Lagrange’s equations with the set of four dependent generalized coordinates – angles $\phi$ and $\theta$, and disk-fixed coordinates $y_G$ and $z_G$ of $G$ relative to $O$. The system consists of a disk $D$ of mass $m_d$ and radius $R$, and a uniform slender bar $B$ of mass $m$ and length $\ell$. The angle $\phi$ describes the rotation of the disk about the $Z$ axis, and the angle $\theta$ describes the rotation of the bar $B$ about the $X'$ axis. A linear, torsional spring-damper is located between $B$ and $D$ at pin $P$. The torsional spring has stiffness $k$ and is unstretched when $\theta = 0$. The torsional damper has coefficient $c$. A motor torque $M_\phi$ is applied to the disk about the $Z$ axis, and a motor torque $M_\theta$ is applied to $B$ about the $X'$ axis.

Answers:

Equations of Motion

\[
\left(\frac{1}{2}m_d R^2\right)\ddot{\phi} + m y_G^2 \ddot{\phi} + 2m y_G \dot{y}_G \dot{\phi} + \frac{m \ell^2}{12} S_\theta \ddot{\phi} + \frac{m \ell^2}{6} S_\theta C_\theta \dot{\phi} \dot{\phi} = M_\phi
\]

\[
\left(\frac{m \ell^2}{12}\right)\ddot{\theta} - \frac{m \ell^2}{12} S_\theta C_\theta \dot{\phi}^2 + c \dot{\theta} + k \theta = M_\theta - \frac{1}{2} \ell C_\theta \lambda_1 - \frac{1}{2} \ell S_\theta \lambda_2
\]

\[
m \dddot{y}_G - m y_G \dddot{\phi}^2 = \lambda_1 \quad m \dddot{z}_G + mg = \lambda_2
\]

Constraint equations

\[
\dddot{y}_G = \left(\frac{1}{2} \ell C_\theta\right)\ddot{\theta} - \left(\frac{1}{2} \ell S_\theta\right)\dddot{\theta}^2
\]

\[
\dddot{z}_G = \left(\frac{1}{2} \ell S_\theta\right)\ddot{\theta} + \left(\frac{1}{2} \ell C_\theta\right)\dddot{\theta}^2
\]

6.5 Spinning Top – Using d’Alembert’s Principle, find the equations of motion of the three degree-of-freedom spinning top shown in the diagram. Assume the moments of inertia of the top about the $\epsilon_1$ and $\epsilon_2$ directions are $I_1 = I_2 = I$, and the moment of inertia about the $\epsilon_3$ direction is $I_3$. Also, assume point $O$ is fixed and acts like a ball-and-socket joint. Use the dependent set of four Euler parameters as generalized coordinates to define the orientation of the top.

\[
\{q\}_{4\times 1} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}^T
\]

The unit vector set $T : (\epsilon_1, \epsilon_2, \epsilon_3)$ is fixed-in and rotates with the top.
Answers:

d’Alembert’s Principle

\[ 2 \varepsilon_4 \left( I + mL^2 \right) \dot{\omega}_1 - 2 \varepsilon_3 \left( I + mL^2 \right) \dot{\omega}_2 + 2 \varepsilon_2 \left( I_3 \right) \dot{\omega}_3 - 2 \varepsilon_3 \left( I + mL^2 - I_3 \right) \omega_1 \omega_3 + 2 \varepsilon_4 \left( I_3 - I - mL^2 \right) \omega_2 \omega_3 = 4mgL \left[ \varepsilon_3 \left( e_3^2 + e_4^2 \right) \right] + \varepsilon_1 \lambda \]

\[ 2 \varepsilon_3 \left( I + mL^2 \right) \dot{\omega}_1 + 2 \varepsilon_4 \left( I + mL^2 \right) \dot{\omega}_2 - 2 \varepsilon_1 \left( I_3 \right) \dot{\omega}_3 + 2 \varepsilon_4 \left( I + mL^2 - I_3 \right) \omega_1 \omega_3 + 2 \varepsilon_3 \left( I_3 - I - mL^2 \right) \omega_2 \omega_3 = 4mgL \left[ \varepsilon_2 \left( e_4^2 + e_3^2 \right) \right] + \varepsilon_2 \lambda \]

\[-2 \varepsilon_2 \left( I + mL^2 \right) \dot{\omega}_1 + 2 \varepsilon_1 \left( I + mL^2 \right) \dot{\omega}_2 + 2 \varepsilon_4 \left( I_3 \right) \dot{\omega}_3 + 2 \varepsilon_1 \left( I + mL^2 - I_3 \right) \omega_1 \omega_3 + 2 \varepsilon_2 \left( I + mL^2 - I_3 \right) \omega_2 \omega_3 = -4mgL \left[ \varepsilon_3 \left( e_1^2 + e_2^2 \right) \right] + \varepsilon_3 \lambda \]

Kinematical differential equations (which includes the constraint equation):

\[
\begin{bmatrix}
\dot{\varepsilon}_1 \\
\dot{\varepsilon}_2 \\
\dot{\varepsilon}_3 \\
\dot{\varepsilon}_4
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
\varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\
\varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\
-\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\
-\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
0
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
\varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\
\varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\
-\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\
-\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4
\end{bmatrix}
\]

6.6 Given the equations of motion of the spinning top of Exercise 6.5, show that the value of the Lagrange multiplier is \( \lambda = 0 \). Hint: This can be done by multiplying the first equation by \( \varepsilon_1 \), the second by \( \varepsilon_2 \), the third by \( \varepsilon_3 \), and the fourth by \( \varepsilon_4 \), and then adding the equations. A similar process can be used to show the equations of motion of Exercise 6.5 are equivalent to those found in Exercise 5.6 of this volume.

6.7 Spinning Top – Using d’Alembert’s Principle, find the equations of motion of the three degree-of-freedom spinning top shown in the diagram. Assume the moments of inertia of the top about the \( \varepsilon_1 \) and \( \varepsilon_2 \) directions are \( I_1 = I_2 = I \), and the moment of inertia about the \( \varepsilon_3 \) direction is \( I_3 \). Also, assume point O is fixed and acts like a ball in socket joint. Use the following dependent set of generalized coordinates – the base-frame coordinates of G to define its location relative to O, and Euler parameters to define the orientation of the top. That is,

\[
\{q\}_q = \begin{bmatrix}
x_1 & x_2 & x_3 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4
\end{bmatrix}^T
\]
The unit vector set $T : (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is fixed-in and rotates with the top.

Answers:

Position coordinates:
\[
\begin{align*}
 m\ddot{x}_1 &= \dot{\lambda}_1 \\
 m\ddot{x}_2 &= \dot{\lambda}_2 \\
 m\ddot{x}_3 &= -mg + \dot{\lambda}_3
\end{align*}
\]

Euler parameters:
\[
\begin{align*}
 2\varepsilon_4 \left[ I\dot{\omega}_1 + (I_3-I)\omega_2\omega_3 \right] - 2\varepsilon_3 \left[ I\dot{\omega}_2 + (I-I_3)\omega_1\omega_3 \right] + 2\varepsilon_2 I_3\dot{\omega}_3 &= \\
 2\varepsilon_3 \left[ I\dot{\omega}_1 + (I_3-I)\omega_2\omega_3 \right] + 2\varepsilon_4 \left[ I\dot{\omega}_2 + (I-I_3)\omega_1\omega_3 \right] - 2\varepsilon_2 I_3\dot{\omega}_3 &= \\
 -2\varepsilon_2 \left[ I\dot{\omega}_1 + (I_3-I)\omega_2\omega_3 \right] + 2\varepsilon_1 \left[ I\dot{\omega}_2 + (I-I_3)\omega_1\omega_3 \right] + 2\varepsilon_4 I_3\dot{\omega}_3 &= \\
 -2\varepsilon_1 \left[ I\dot{\omega}_1 + (I_3-I)\omega_2\omega_3 \right] - 2\varepsilon_2 \left[ I\dot{\omega}_2 + (I-I_3)\omega_1\omega_3 \right] - 2\varepsilon_4 I_3\dot{\omega}_3 &= \\
\end{align*}
\]

Kinematical Differential Equations and Constraint Equations:
\[
\begin{bmatrix}
 \dot{\mathbf{e}}_1 \\
 \dot{\mathbf{e}}_2 \\
 \dot{\mathbf{e}}_3 \\
 \dot{\mathbf{e}}_4
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
 \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\
 \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\
 -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\
 -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4
\end{bmatrix} \begin{bmatrix}
 \omega_1 \\
 \omega_2 \\
 \omega_3 \\
 0
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
 \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\
 \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\
 -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\
 -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4
\end{bmatrix} \begin{bmatrix}
 \omega_1 \\
 \omega_2 \\
 \omega_3 \\
 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
 \dot{x}_1 - 2L(\varepsilon_3\dot{\mathbf{e}}_1 + \varepsilon_4\dot{\mathbf{e}}_2 + \varepsilon_1\dot{\mathbf{e}}_3 + \varepsilon_2\dot{\mathbf{e}}_4) \\
 \dot{x}_2 - 2L(-\varepsilon_4\dot{\mathbf{e}}_1 + \varepsilon_3\dot{\mathbf{e}}_2 + \varepsilon_2\dot{\mathbf{e}}_3 - \varepsilon_1\dot{\mathbf{e}}_4) \\
 \dot{x}_3 + 2L(\varepsilon_1\dot{\mathbf{e}}_1 + \varepsilon_2\dot{\mathbf{e}}_2 - \varepsilon_3\dot{\mathbf{e}}_3 - \varepsilon_4\dot{\mathbf{e}}_4)
\end{bmatrix} = \begin{bmatrix}
 0 \\
 0 \\
 0
\end{bmatrix}
\]
References:

Addendum – The Constraint Relaxation Method: Meaning of Lagrange Multipliers

Previously, it was noted that if a dynamic system is described using generalized coordinates \( q_k \) \((k = 1, \ldots, n)\), and if the system is subjected to a set of independent constraint equations of the form

\[
\sum_{k=1}^{n} a_{jk} \dot{q}_k + a_{j0} = 0 \quad (j = 1, \ldots, m)
\]

then, Lagrange’s equations of motion can be written using Lagrange multipliers as

\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k = 1, \ldots, n)
\]

or

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \left( F_{q_k} \right)_{nc} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k = 1, \ldots, n)
\]

Here, \( K \) is the kinetic energy of the system, \( F_{q_k} \) is the generalized force associated with the generalized coordinate \( q_k \), \( L \) is the Lagrangian of the system, \( V \) is the potential energy function for the conservative forces and torques, \( \left( F_{q_k} \right)_{nc} \) is the generalized force associated with \( q_k \) for the non-conservative forces and torques, only, \( \lambda_j \) is the Lagrange multiplier associated with the \( j^{th} \) constraint equation, and \( a_{jk} \) \((j = 1, \ldots, m; \ k = 1, \ldots, n)\) are the coefficients from the constraint equations.

Alternatively, some or all the constraints can be relaxed (or removed) and replaced with unknown force and/or torque components that are required to maintain the constraints. Then, formulate the equations of motion including the effects of the constraint forces. Using this approach, the two forms of Lagrange’s equations can be written as

\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \left( F_{q_k} \right)_{\text{constraints}} \quad (k = 1, \ldots, n)
\]

and

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \left( F_{q_k} \right)_{nc} + \left( F_{q_k} \right)_{\text{constraints}} \quad (k = 1, \ldots, n)
\]

Comparing Lagrange’s equations from the two approaches, it is clear that

\[
\left( F_{q_k} \right)_{\text{constraints}} = \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k = 1, \ldots, n)
\]
It is clear from this last result that the Lagrange multipliers are directly related to the forces and/or torques required to maintain the constraints.

**Example: The Simple Pendulum of Example 1**

Consider the simple pendulum of Example 1 with coordinates \((x, y)\) to describe the position of the mass \(m\). Then relax the length constraint of the pendulum and replace the light connecting rod with a force \(T\) which is directed from the mass to the external support. Using this approach, Lagrange's equations can be written in the form

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q_k} \right) - \frac{\partial L}{\partial q_k} = \left( F_{q_k} \right)_{\text{zero}} + \left( F_{q_k} \right)_{\text{constraint}} = \left( F_{q_k} \right)_{\text{constraint}}
\]

Where the Lagrangian is \(L = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) + mg\), and the contributions of the constraint force on the right sides of the equations are

\[
\left( F_{x} \right)_{\text{constraint}} = T \cdot \left( \frac{\partial y}{\partial x} \right) = T \left( -\frac{x}{L} \hat{i} - \frac{y}{L} \hat{j} \right) \cdot \left( \dot{x} \hat{i} + \dot{y} \hat{j} \right) \frac{\partial \dot{x}}{\partial \dot{x}} = -T \left( x/L \right)
\]

\[
\left( F_{y} \right)_{\text{constraint}} = T \cdot \left( \frac{\partial y}{\partial y} \right) = T \left( -\frac{x}{L} \hat{i} - \frac{y}{L} \hat{j} \right) \cdot \left( \dot{x} \hat{i} + \dot{y} \hat{j} \right) \frac{\partial \dot{y}}{\partial \dot{y}} = -T \left( y/L \right)
\]

Substituting into Lagrange's equations and supplementing with the twice differentiated constraint equation gives the following equations of motion

\[
\begin{align*}
m\ddot{x} + \left( \frac{x}{L} \right) T &= 0 \\
m\ddot{y} - mg + \left( \frac{y}{L} \right) T &= 0 \\
x\dddot{x} + y\dddot{y} + \dot{x}^2 + \dot{y}^2 &= 0
\end{align*}
\]

In Example 1 the equations of motion for this system were written using Lagrange multipliers as follows.

\[
\begin{align*}
m\ddot{x} - \lambda x &= 0 \\
m\ddot{y} - mg - \lambda y &= 0 \\
x\dddot{x} + y\dddot{y} + \dot{x}^2 + \dot{y}^2 &= 0
\end{align*}
\]

Comparing these results, \(\lambda = -T/L\). Hence, the Lagrange multiplier is related to the force per unit length in the light connecting rod.