

Introductory Control Systems

Second-Order System Step Response: Summary

Ref: R. N. Clark, *Introduction to Automatic Control Systems*, John Wiley & Sons, Inc, 1962
 Dorf & Bishop, *Modern Control Systems*, 12th edition, Prentice-Hall, Inc, 2010

The transfer functions for second-order systems can be written in one of the two general forms (depending on whether the system has a zero or not)

$$\text{Case 1: } \frac{Y}{R}(s) = \frac{q}{s^2 + p s + q}$$

$$\text{Case 2: } \frac{X}{R}(s) = \frac{(q/a)(s+a)}{s^2 + p s + q}$$

Both cases can be broken into different types of response depending on whether the poles of the system are **real and unequal**, **real and equal**, **complex**, or **purely imaginary**. The discussion below considers only the response of **stable** systems. Stable systems (as defined here) are systems whose poles have **non-positive** real parts.

Case 1: There are **four types of motion** that are possible in this case. $\frac{Y}{R}(s) = \frac{q}{s^2 + p s + q}$

- If the poles are **real and unequal**, the response is **over-damped** with **no overshoot**. If the poles are **widely separated**, the response may be **dominated** by the smaller (**slower**) pole.
- If the poles are **real and equal**, the response is **critically damped** with **no overshoot**.
- If the poles are **complex** the system is **under-damped** with an overshoot.
- If the poles are **purely imaginary**, then the system has **no damping** and the response will be **oscillatory** with no reduction in amplitude. The response is said to be harmonic.

Fig. 1 shows the step response of a system with **real, unequal** poles, and Fig. 2 shows the step response of a system with **complex** poles. Both plots show the 2% **settling time** of the system, and Fig. 2 shows the **percent overshoot** as well.

The system of Fig. 1 is an **over-damped** system with poles at $s = -2$ and $s = -5$. The **settling times** of the two poles are $4/2 = 2$ seconds and $4/5 = 0.8$ seconds. From the step response plot, it is clear the system settling time is **close** to that of the **slower pole**.

The system of Fig. 2 is an **under-damped** system with **natural frequency** $\omega_n = \sqrt{25} = 5$ (rad/s) and a **damping ratio** of $\zeta = 5/2\omega_n = 0.5$. The measured settling time of 1.62 seconds is consistent with the estimated settling time of $T_s = 4/\zeta\omega_n = 4/(5/2) = 8/5 \approx 1.6$ (sec).

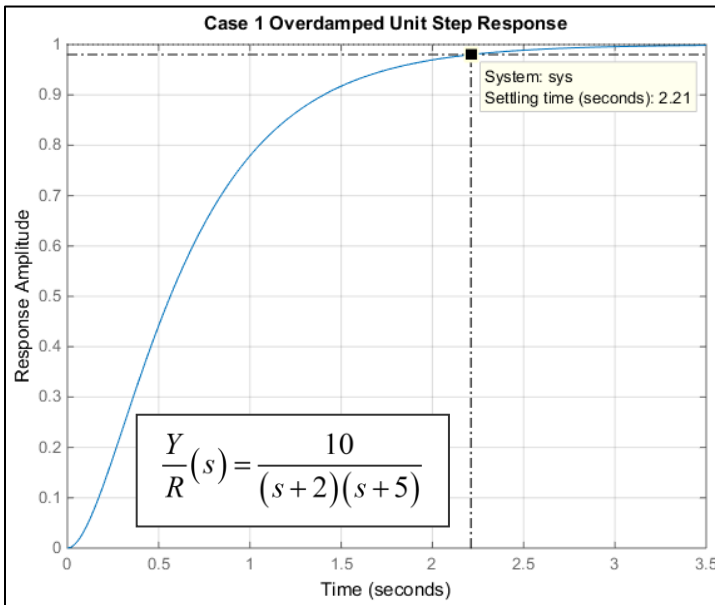


Fig. 1 Case 1 Second Order, *Over-damped* Unit Step Response

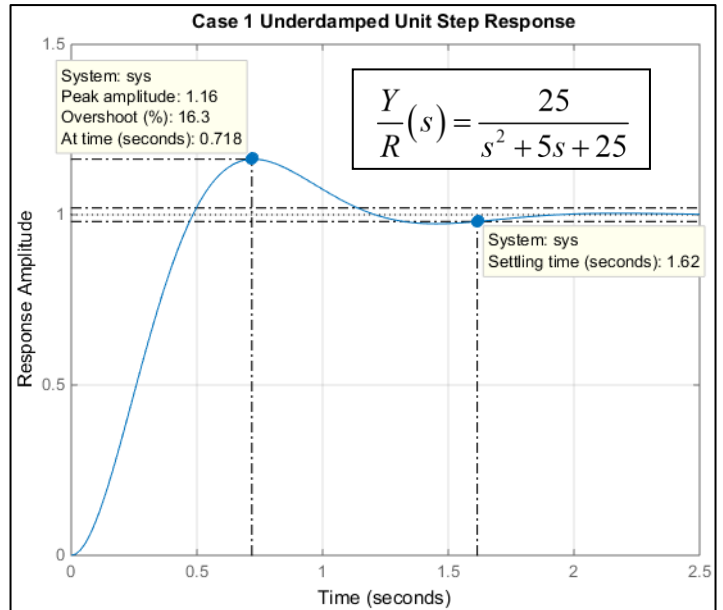


Fig. 2 Case 1 Second Order, *Under-damped* Unit Step Response

The measured percent overshoot of 16.3% is consistent with plot of percent overshoot versus damping ratio presented in earlier notes and Fig. 5.8 of the Dorf & Bishop text. The plot from previous notes is reproduced here for convenience of the reader. See Fig. 3 below.

Fig. 4 shows Case 1 second-order system unit step responses for *various values* of the *damping ratio* ζ . The natural frequency of the system is $\omega_n = 5$ (rad/s). Note as the *damping ratio* gets *smaller*, the responses become *more oscillatory* with *larger overshoots*.

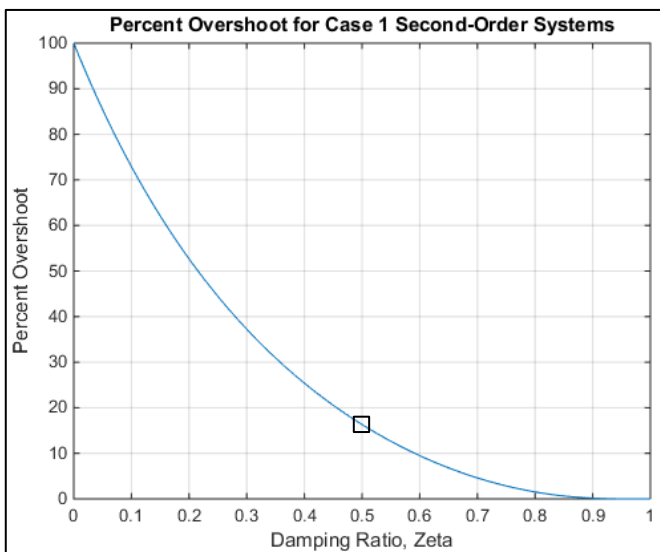


Fig. 3 Percent Overshoot vs. Damping Ratio ζ

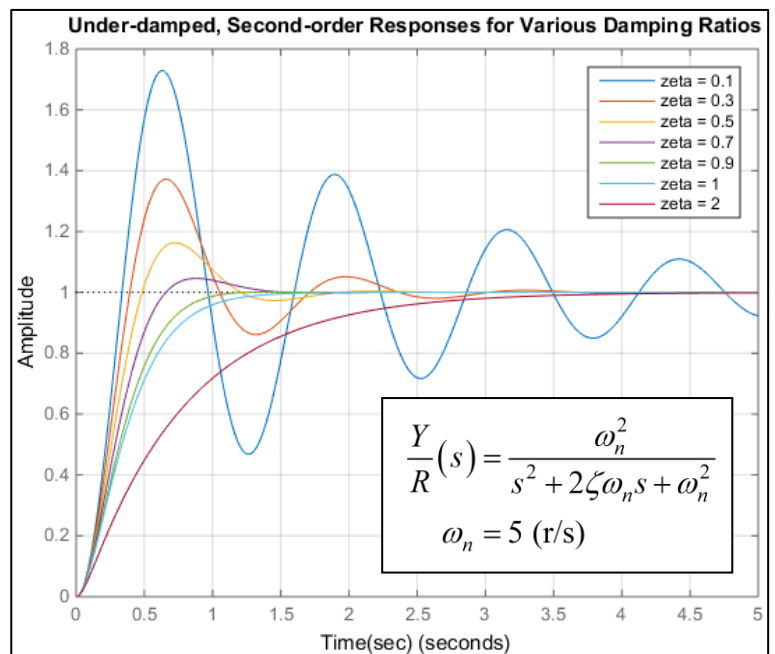


Fig. 4 Second Order, Case 1, Unit Step Responses

Case 2: $\frac{X}{R}(s) = \frac{(q/a)(s+a)}{s^2 + ps + q}$

- The motion of Case 2 systems is complicated somewhat over that of Case 1 systems by the presence of the **zero** ($s+a$) in the transfer function. The **zero** has *little effect* on the **settling time** of the system but *can significantly affect the overshoot*.
- How *much effect* the zero has on overshoot depends on *where* it is (along the real axis) compared to the poles of the system.
 - **Zero is far to the left of the poles:** it has *little effect* on the response of the system. This makes the response much like that of a Case 1 system.
 - **Zero located near or inside the poles:** it will *significantly affect* the overshoot of the system.

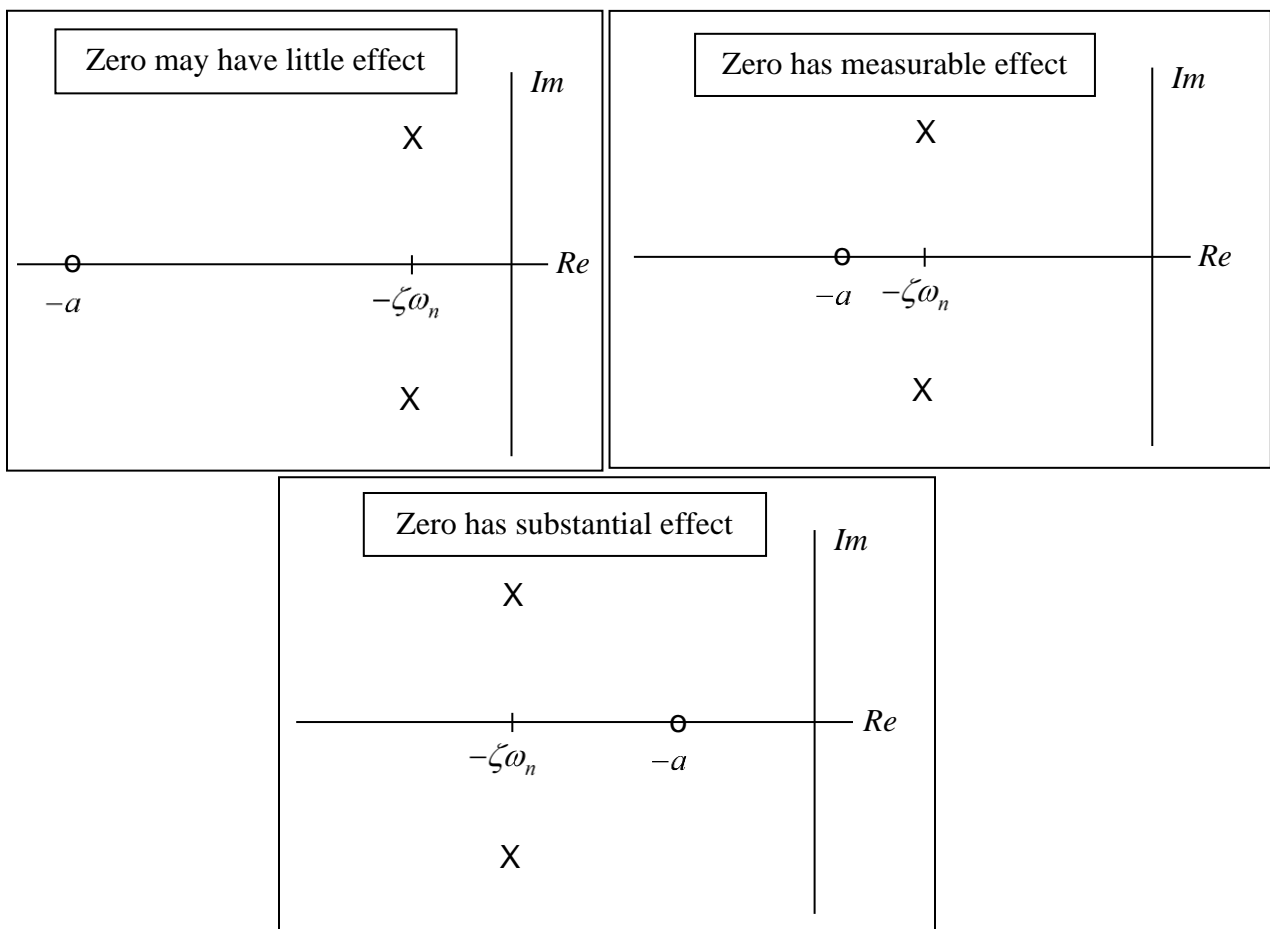


Fig. 4 below shows the step response of a Case 2 **over-damped**, second order system. The system is the same as that shown in Fig. 1, except it has a zero at $s = -1$. Even though the system is over-damped, the presence of the zero *causes significant overshoot*. The settling time of the system is unchanged.

Fig. 5 below shows the step response of a Case 2 **under-damped**, second order system. The system is the same as that of Fig. 2, except it has a zero at $s = -1$. The zero *increases the overshoot* of the system from 16% to over

200%. The settling time has increased (somewhat) to 1.9 seconds. Note that the system narrowly escapes the 2% band at just about 1.6 seconds which is the settling time for the corresponding Case 1 system.

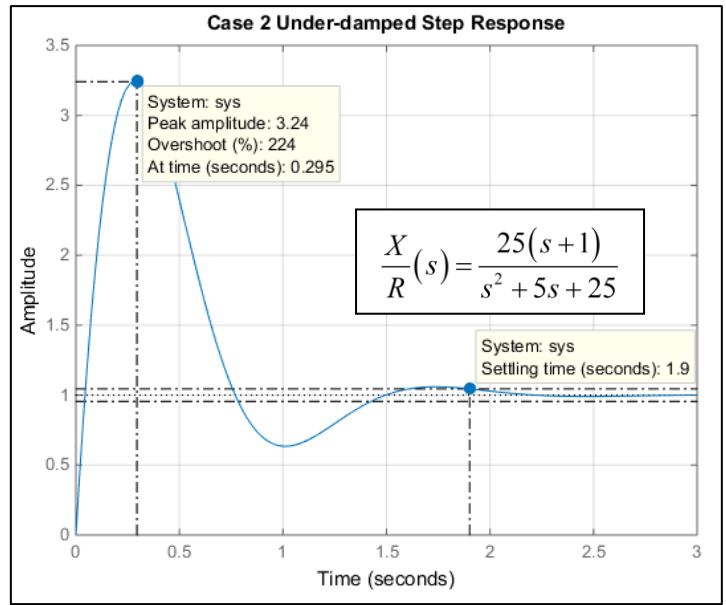
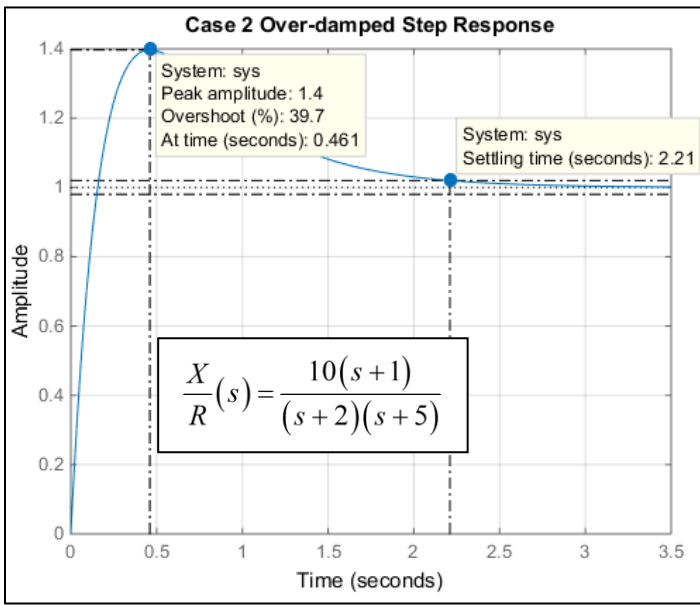


Fig. 4 Case 2 Second Order, Over-damped Step Response

Fig. 5 Case 2 Second Order, Under-damped Step Response

The effects of the presence of a zero on the percent overshoot for under-damped and critically damped systems can be estimated using Fig. 6 below that was developed in previous notes. Parameter $\beta \triangleq a/\zeta\omega_n$ for the under-damped systems and $\beta \triangleq a/\alpha$ for the critically damped systems (repeated poles at $s = -\alpha$). This plot is similar to Fig. 5.13 in the Dorf and Bishop text.

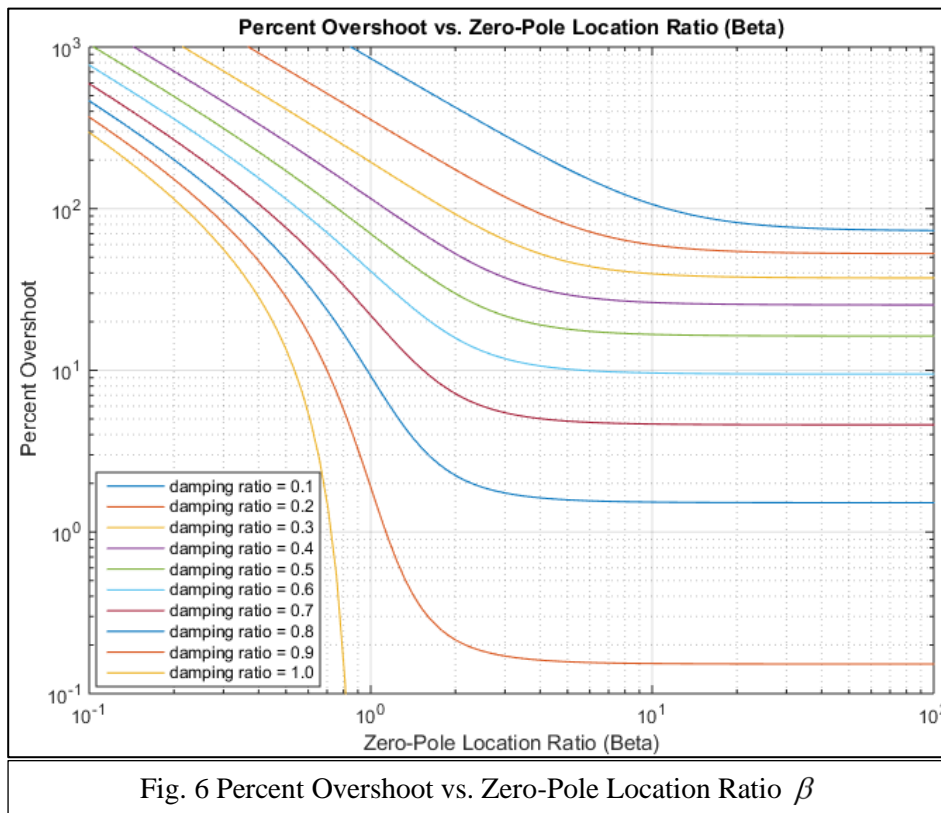


Fig. 6 Percent Overshoot vs. Zero-Pole Location Ratio β