Introductory Control Systems
Second-Order System Response – Theoretical Analysis

Reference: R.N. Clark, *Introduction to Automatic Control Systems*, John Wiley & Sons, 1962.

The transfer functions for normalized second-order systems fall into the following two general categories.

Case 1: $\boxed{\frac{Y}{R}(s)=\frac{q}{s^2+p\,s+q}}$ Case 2: $\boxed{\frac{X}{R}(s)=\frac{(q/a)(s+a)}{s^2+p\,s+q}}$

For the purpose of the analysis that follows, it is assumed the ***poles*** and ***zero*** of these transfer functions are in the ***left-half*** of the *s*-plane. Note here the classifications of Case 1 and Case 2 are not universally accepted but are used here for convenience. The analysis that follows develops formulae for T_p the peak-time and $\%OS$ the percent overshoot of these systems associated with a unit step input.

Case 1 Under-damped Systems:

The general form of the ***normalized*** transfer functions of ***under-damped***, ***second-order systems*** with a constant numerator is

$$\boxed{\frac{Y}{R}(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

Letting $R(s)=\frac{1}{s}$ for a unit-step input and using Laplace transform tables, the response function $y(t)$ can be written as follows.

$$\boxed{y(t)=1-\left(\frac{1}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\,\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t+\phi\right)} \qquad \boxed{\phi=\cos^{-1}(\zeta)} \qquad 0<\zeta<1 \qquad (1)$$

An example system response is shown in Fig. 1 for systems with natural frequency $\omega_n=5\text{ (rad/s)}$ and damping ratio $\zeta=0.5$. As expected, the response starts at zero and reaches a final value of one. In the transition from initial to final values, the system overshoots and oscillates about the final value. The time at which the system reaches a maximum value just after its first crossing of the final value is known as the “peak-time”. The percent overshoot of the system is measured at this time. As indicated on the plot, the peak-time and percent overshoot for this system are $T_p\approx 0.718\text{ (sec)}$ and $\%OS\approx 16.3\%$.

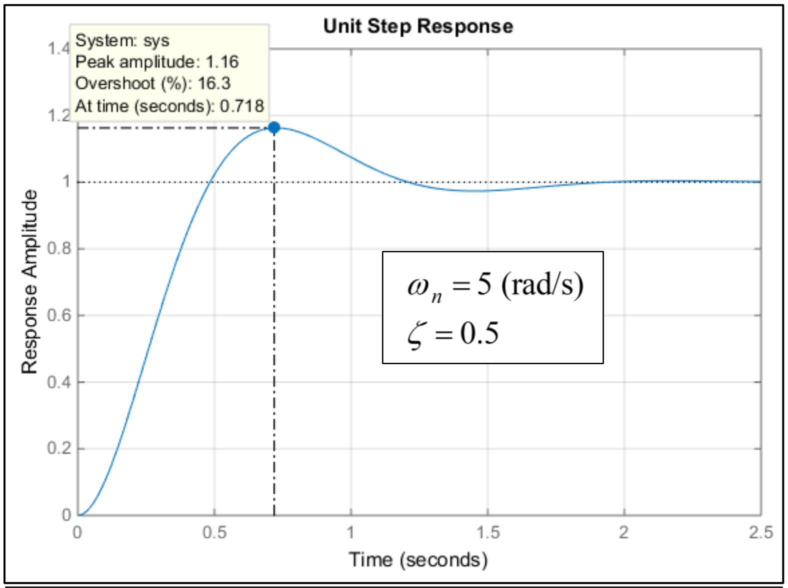

Fig. 1 Step Response of an Example Case 1 System

Formulae for the peak-time and percent overshoot for these types of systems can be derived using the response function of Eq. (1). First, the peak-time is found by finding the times when the derivative of $y(t)$ is zero. Before differentiating, $y(t)$ is expanded using the trigonometric identity

$$\boxed{\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}$$

Using this identity, Eq. (1) can be rewritten as follows.

$$y(t)=1-\left(\frac{1}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\left[\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\cos(\phi)+\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\sin(\phi)\right]$$

Note from Eq. (1) that $\cos(\phi)=\zeta$ and, consequently, $\sin(\phi)=\sqrt{1-\cos^2(\phi)}=\sqrt{1-\zeta^2}$. Substituting these results into the above equation gives

$$\boxed{y(t)=1-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)-e^{-(\zeta\omega_n t)}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)} \tag{2}$$

Eq. (2) can now be differentiated and simplified as follows.

$$\dot{y}(t)\triangleq\frac{dy}{dt}=\left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)-\left(\frac{\zeta\,\omega_n\cancel{\sqrt{1-\zeta^2}}}{\cancel{\sqrt{1-\zeta^2}}}\right)e^{-(\zeta\omega_n t)}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$
$$+\,\zeta\omega_n e^{-(\zeta\omega_n t)}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)+\omega_n\sqrt{1-\zeta^2}\,e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$= \left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right) - \cancel{\zeta\,\omega_n e^{-(\zeta\omega_n t)}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)}$$

$$+ \cancel{\zeta\,\omega_n e^{-(\zeta\omega_n t)}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)} + \omega_n\sqrt{1-\zeta^2}\,e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$= \left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right) + \omega_n\sqrt{1-\zeta^2}\,e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$= \left(\frac{\zeta^2\omega_n + \omega_n - \omega_n\zeta^2}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$\Rightarrow \boxed{\dot{y}(t) = \left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)} \qquad (3)$$

Using Eq. (3) it is clear that the derivative of $y(t)$ is zero when the sine function is zero. This occurs at $t = \{0, \pi, 2\pi, \ldots\}$. The peak-time T_p occurs at the first peak after the start of the response. So,

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \quad \Rightarrow \boxed{\omega_n T_p = \frac{\pi}{\sqrt{1-\zeta^2}}} \qquad (4)$$

Using Eqs. (2) and (4) the percent overshoot can now be written as follows.

$$\%OS = 100\left(y(T_p)-1\right) = 100\left(\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n T_p)}\underbrace{\sin(\pi)}_{\text{zero}} - e^{-(\zeta\omega_n T_p)}\underbrace{\cos(\pi)}_{-1}\right)$$

$$\Rightarrow \boxed{\%OS = 100\,e^{-(\zeta\omega_n T_p)} = 100\,e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}} \qquad (5)$$

The functions for $\omega_n T_p$ and $\%OS$ of Eqs. (4) and (5) are plotted in the Figs. 2 and 3 below. Note that they are functions of the damping ratio ζ only. The function for $\omega_n T_p$ starts at a value of 3.142 and increases to infinity as ζ increases from zero to one. The peak-time is infinite for $\zeta = 1$ because the response only approaches the final value as $t \to \infty$, hence theoretically never reaching the final value. The percent overshoot decreases from 100% to zero as ζ increases from zero to one. Using Eq. (5) the percent overshoot for systems with a damping ratio of $\zeta = 0.5$ is calculated to be $\%OS \approx 16.3\%$ which is consistent with the measured results in Fig. 1 above.

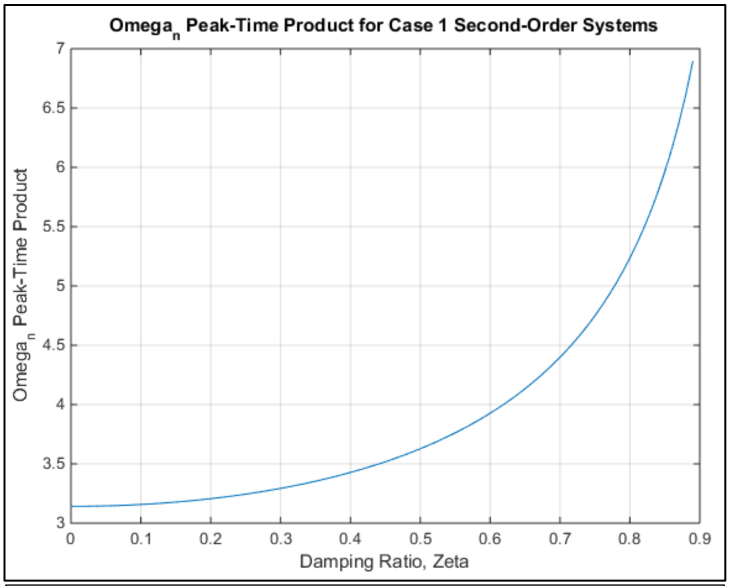

Fig. 2 $\omega_n T_p$ Product vs. Damping Ratio ζ

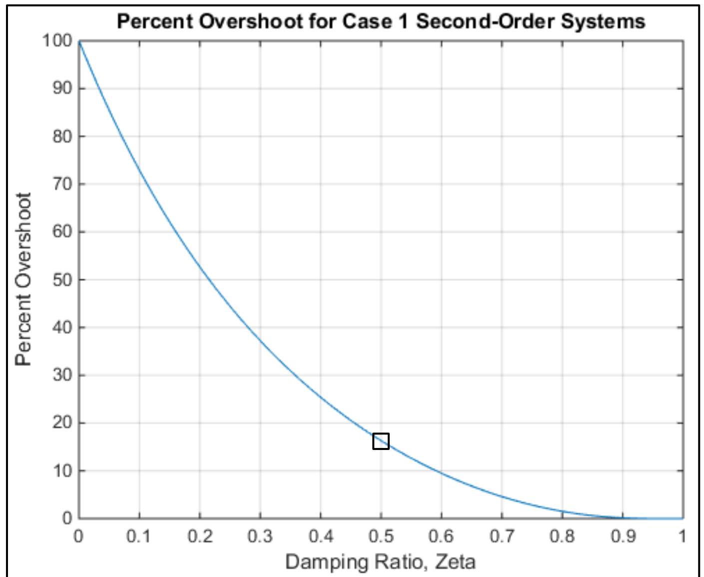

Fig. 3 Percent Overshoot vs. Damping Ratio ζ

Case 2 Under-damped Systems:

The general form of the ***normalized*** transfer functions of ***under-damped***, ***second-order systems*** with a real zero is

$$\frac{X}{R}(s) = \frac{(\omega_n^2/a)(s+a)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{6}$$

An example response is shown here for systems with natural frequency $\omega_n = 5\ \text{(rad/s)}$, damping ratio $\zeta = 0.5$, and $a = 1$. Note the natural frequency and damping ratio are the same as used in the Case 1 example above. As expected, the response starts at zero and reaches a final value of one. As with the Case 1 systems, the system overshoots and oscillates about the final value. The percent overshoot of the Case 2 system, however, is much larger than that of the Case 1 system. As indicated on the plot, the peak-time and percent overshoot for this system are $T_p \approx 0.295\ \text{(sec)}$ and $\%OS \approx 224\%$. So, this Case 2 system has a faster response with more overshoot than the corresponding Case 1 system.

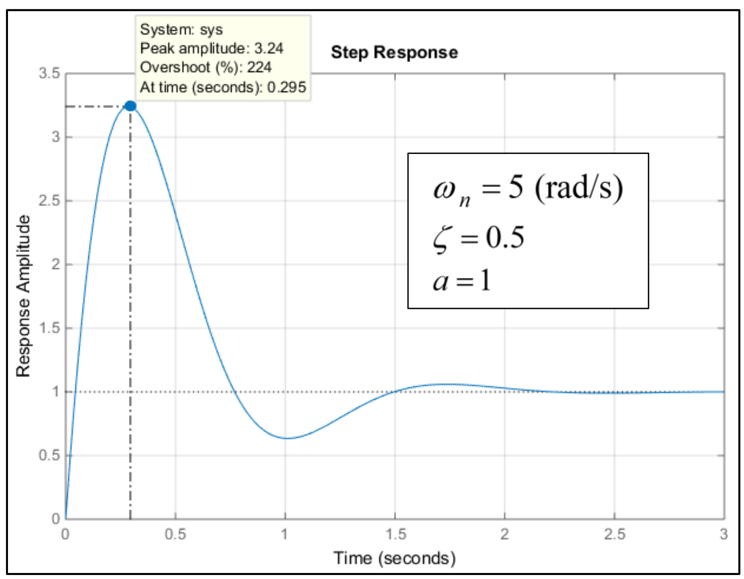

Fig. 4 Step Response of an Example Case 2 System

The response of Case 2 under-damped systems can be studied by first separating the transfer function into the sum of two transfer functions. Using this approach, Eq. (6) can be rewritten as follows.

$$\frac{X}{R}(s)=\frac{(\omega_n^2/a)(s+a)}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{a}\left[\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}\right]+\frac{\omega_n^2}{a}\left[\frac{a}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

$$=\left(\frac{1}{a}\right)s\left[\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right]+\left[\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

$$\Rightarrow \boxed{\frac{X}{R}(s)=\left(\frac{1}{a}\right)\left[\frac{\dot{Y}}{R}(s)\right]+\left[\frac{Y}{R}(s)\right]} \tag{7}$$

In the time domain, Eq. (7) can be written as follows.

$$\boxed{x(t)=y(t)+\left(\tfrac{1}{a}\right)\dot{y}(t)} \tag{8}$$

Eq. (8) shows the response of Case 2 systems are the sum of the response of the corresponding Case 1 system, that is $y(t)$, plus $1/a$ times the derivative of $y(t)$. For small values of a the response of Case 2 systems will be significantly different than the corresponding Case 1 system; however, for large values of a the response of Case 2 systems should be very similar to the corresponding Case 1 system. Using Eq. (8) the peak-time and percent overshoot of Case 2 under-damped systems are derived below.

Substituting from Eqs. (2) and (3) into Eq. (8), the response of Case 2 systems can be written as follows.

$$x(t)=1-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)-e^{-(\zeta\omega_n t)}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$+\left(\frac{1}{a}\right)\left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right)e^{-(\zeta\omega_n t)}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

Or,

$$\boxed{x(t)=1+e^{-(\zeta\omega_n t)}\left[\left(\frac{(\omega_n/a)-\zeta}{\sqrt{1-\zeta^2}}\right)\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)-\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\right]} \tag{9}$$

Using Eq. (9), the derivative of $x(t)$ can be calculated as follows.

$$\dot{x}(t)=-\zeta\omega_n e^{-(\zeta\omega_n t)}\left[\left(\frac{(\omega_n/a)-\zeta}{\sqrt{1-\zeta^2}}\right)\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)-\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\right]$$

$$+e^{-(\zeta\omega_n t)}\left[\left(\frac{(\omega_n/a)-\zeta}{\cancel{\sqrt{1-\zeta^2}}}\right)\omega_n\cancel{\sqrt{1-\zeta^2}}\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)+\omega_n\sqrt{1-\zeta^2}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\right]$$

$$= e^{-(\zeta\omega_n t)}\left[\omega_n\sqrt{1-\zeta^2}+\left(\frac{-\zeta\omega_n\left(\omega_n/a\right)+\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\right)\right]\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$+e^{-(\zeta\omega_n t)}\left[\cancel{\zeta\omega_n}+\omega_n\left(\omega_n/a\right)-\cancel{\omega_n\zeta}\right]\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$= e^{-(\zeta\omega_n t)}\left[\frac{\omega_n\left(1-\cancel{\zeta^2}\right)-\zeta\omega_n\left(\omega_n/a\right)+\cancel{\zeta^2\omega_n}}{\sqrt{1-\zeta^2}}\right]\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$+e^{-(\zeta\omega_n t)}\left(\omega_n^2/a\right)\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

$$\Rightarrow \boxed{\dot{x}(t)=\omega_n e^{-(\zeta\omega_n t)}\left[\left(\frac{1-\left(\zeta\omega_n/a\right)}{\sqrt{1-\zeta^2}}\right)\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)+\left(\omega_n/a\right)\cos\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)\right]} \qquad (10)$$

This result can be written as a single phase-shifted sine function. In this process the variable $\beta \triangleq a/\zeta\omega_n$ which indicates the relative location of the zero and the complex poles along the real axis is introduced.

$$\boxed{\dot{x}(t)=\omega_n M_2 e^{-\zeta\omega_n t}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t+\psi_2\right)} \qquad (11)$$

with

$$M_2^2=\left(\frac{1-\left(\zeta\omega_n/a\right)}{\sqrt{1-\zeta^2}}\right)^2+\left(\omega_n/a\right)^2=\frac{1-2\left(\zeta\omega_n/a\right)+\cancel{\left(\zeta\omega_n/a\right)^2}+\left(\omega_n/a\right)^2\left(1-\cancel{\zeta^2}\right)}{\left(1-\zeta^2\right)}$$

$$\Rightarrow \boxed{M_2\left(\beta,\zeta\right)=\frac{\sqrt{1-2\left(\zeta\omega_n/a\right)+\left(\zeta\omega_n/a\right)^2/\zeta^2}}{\sqrt{1-\zeta^2}}=\frac{\sqrt{1-(2/\beta)+(1/\beta^2\zeta^2)}}{\sqrt{1-\zeta^2}}} \qquad (12)$$

$$\tan\psi_2=\frac{\left(\omega_n/a\right)}{\left(\frac{1-\left(\zeta\omega_n/a\right)}{\sqrt{1-\zeta^2}}\right)}=\frac{(1/\zeta)\left(\zeta\omega_n/a\right)\sqrt{1-\zeta^2}}{1-\left(\zeta\omega_n/a\right)}=\frac{(1/\beta\zeta)\sqrt{1-\zeta^2}}{1-(1/\beta)} \quad \Rightarrow \boxed{\psi_2\left(\beta,\zeta\right)=\tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta\left(\beta-1\right)}\right]} \qquad (13)$$

Using the results of Eqs. (11) to (13), the times when $\dot{x}(t)$ is zero are seen to be those when the argument of the sine function is a multiple of π. That is,

$$\boxed{\left(\omega_n\sqrt{1-\zeta^2}\right)t+\psi_2=\left\{\pi,\,2\pi,\,\ldots\right\}}$$

The peak-time of $x(t)$ is then given to be

$$\boxed{T_p=\frac{\pi-\psi_2}{\omega_n\sqrt{1-\zeta^2}}} \text{ with } \boxed{\psi_2\left(\beta,\zeta\right)=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta\left(\beta-1\right)}\right)} \qquad (14)$$

Using Eqs. (9) and (14) the percent overshoot of the system can then be written as follows.

$$\%OS = 100\left(x(T_p)-1\right)$$

$$= 100\, e^{-\zeta(\pi-\psi_2)/\sqrt{1-\zeta^2}}\left[\left(\frac{(\omega_n/a)-\zeta}{\sqrt{1-\zeta^2}}\right)\sin\left(\pi-\psi_2\right)-\cos\left(\pi-\psi_2\right)\right]$$

$$= 100\, e^{-\zeta(\pi-\psi_2)/\sqrt{1-\zeta^2}}\left[\left(\frac{(\zeta\omega_n/a)-\zeta^2}{\zeta\sqrt{1-\zeta^2}}\right)\sin\left(\pi-\psi_2\right)-\cos\left(\pi-\psi_2\right)\right]$$

$$\Rightarrow \boxed{\%OS = 100\, e^{-\zeta(\pi-\psi_2)/\sqrt{1-\zeta^2}}\left[\left(\frac{(1/\beta)-\zeta^2}{\zeta\sqrt{1-\zeta^2}}\right)\sin\left(\pi-\psi_2\right)-\cos\left(\pi-\psi_2\right)\right]} \qquad (15)$$

The figure below shows the percent overshoot of Case 2 under-damped systems as a function the zero-complex pole location ratio β for damping ratios from 0.1 to 0.9. Note that unlike Case 1 systems (see Fig. 3), the percent overshoot of Case 2 systems can be well above 100%. The largest percent overshoots occur for the smallest values of β, that is when the zero is located to the right of the complex poles in the *s*-plane. The percent overshoots decrease as the zero is moved farther and farther into the left-half plane.

Note that the curve for each damping ratio reaches a horizontal asymptote as β increases. The percent overshoots associated these asymptotes are the same as those provided for Case 1 systems as shown above in Fig. 3. So, the effect of the zero on the percent overshoot of the system is lessened as it is located farther and farther into the left-half of the *s*-plane relative to the complex poles. Note finally that this figure predicts over 200% overshoot for Case 2 systems with damping ratio $\zeta = 0.5$ and zero-pole location ratio $\beta = 0.4$. See boxed result in Fig. 5.

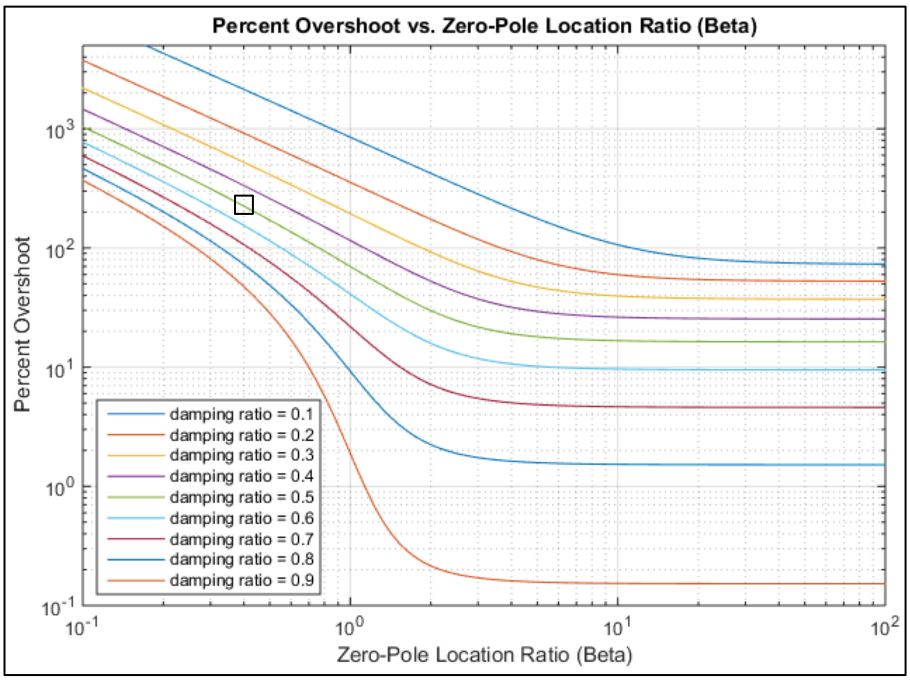

Fig. 5 Percent Overshoot vs. Zero-Pole Location Ratio β

The results in Fig. 5 are consistent with the results presented in Fig. 4 for the example Case 2 under-damped system. They are also consistent with those presented in Fig. 4.44 of the text *Introduction to Automatic Control Systems* by R.N. Clark (John Wiley & Sons, Inc., 1962).

Case 2 Critically Damped Systems:

Eqs. (14) and (15) are singular when $\zeta = 1$, so critically damped systems need to be analyzed separately. The peak time and percent overshoot for critically damped systems can be found using the same process outlined above for under-damped systems. To this end, note the general form of the ***normalized*** transfer functions of ***critically damped***, ***second-order systems*** with a real zero can be written as follows.

$$\boxed{\frac{X}{R}(s) = \left(\frac{\alpha^2}{a}\right)\frac{s+a}{(s+\alpha)^2}} \tag{16}$$

The unit step response of an example critically damped system with $\alpha = 10$ and $a = 4$ is shown in Fig. 6. Case 1 critically damped systems have no overshoot, but as seen in this example, Case 2 critically damped systems can have overshoot. The measured peak time and percent overshoot for this system are $T_p \approx 0.17$ (sec) and $\%OS \approx 28.3\%$. Formulae for the peak time and percent overshoot for Case 2 critically damped systems are derived in the following paragraphs.

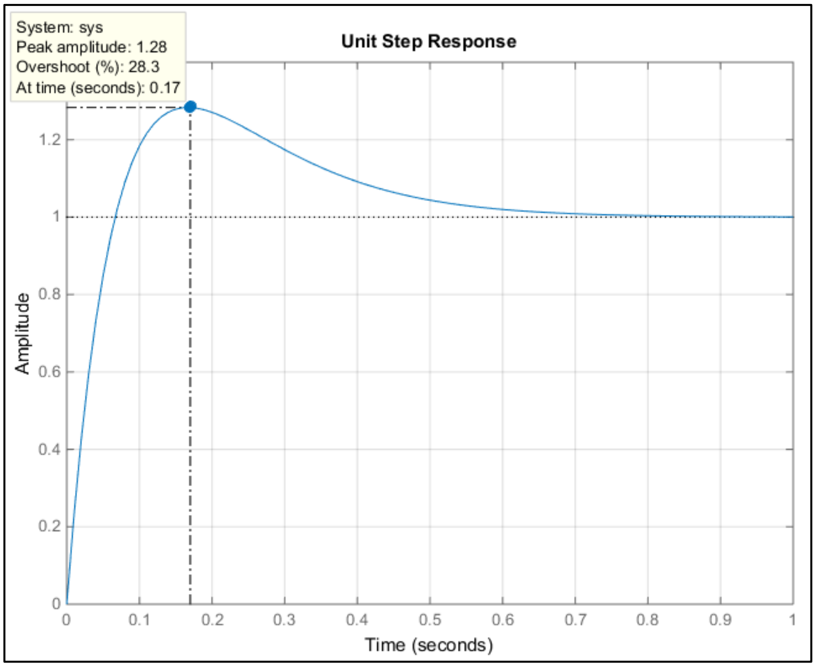

Fig. 6 Step Response of Example Case 2 Critically Damped System

As with under-damped systems, the response of Case 2 critically damped systems can be separated into two parts as follows.

$$\boxed{\frac{X}{R}(s) = \left(\frac{\alpha^2}{a}\right)\left[\frac{s+a}{(s+\alpha)^2}\right] = \left(\frac{1}{a}\right)s\left[\frac{\alpha^2}{(s+\alpha)^2}\right] + \left[\frac{\alpha^2}{(s+\alpha)^2}\right] = \left(\frac{1}{a}\right)\left[\frac{\dot{Y}}{R}(s)\right] + \left[\frac{Y}{R}(s)\right]} \tag{17}$$

Here, as before, $Y(s)$ refers to the Laplace transform of the response of the corresponding Case 1 system. Letting $R(s) = \frac{1}{s}$ for a unit-step input and using Laplace transform tables, the response function $y(t)$ can be written as

$$\boxed{y(t) = 1 - e^{-\alpha t} - \alpha t e^{-\alpha t}} \tag{18}$$

Differentiating this result gives

$$\dot{y}(t) = \cancel{\alpha e^{-\alpha t}} - \cancel{\alpha e^{-\alpha t}} + \alpha^2 t e^{-\alpha t} \quad \Rightarrow \boxed{\dot{y}(t) = \alpha^2 t e^{-\alpha t}} \tag{19}$$

Using Eqs. (18) and (19) the step response of Case 2 critically damped systems can be written as follows.

$$x(t)=y(t)+\tfrac{1}{a}\dot{y}(t)=\left(1-e^{-\alpha t}-\alpha t e^{-\alpha t}\right)+\tfrac{1}{a}\left(\alpha^2 t e^{-\alpha t}\right)\quad\Rightarrow\quad\boxed{x(t)=1-e^{-\alpha t}+\left(\frac{\alpha^2-a\alpha}{a}\right)te^{-\alpha t}} \tag{20}$$

Setting the derivative of $x(t)$ to zero gives

$$\dot{x}(t)=\alpha e^{-\alpha t}+\left(\frac{\alpha^2-a\alpha}{a}\right)e^{-\alpha t}-\alpha\left(\frac{\alpha^2-a\alpha}{a}\right)te^{-\alpha t}=\left[\alpha+\left(\frac{\alpha^2-a\alpha}{a}\right)-\alpha\left(\frac{\alpha^2-a\alpha}{a}\right)t\right]e^{-\alpha t}=0$$

Because $e^{-\alpha t}\neq 0$, the term in square brackets must be zero. So,

$$\boxed{\alpha+\left(\frac{\alpha^2-a\alpha}{a}\right)=\left(\frac{\cancel{a\alpha}+\alpha^2-\cancel{a\alpha}}{a}\right)=\frac{\alpha^2}{a}}\quad=\quad\boxed{\alpha\left(\frac{\alpha^2-a\alpha}{a}\right)T_p=\alpha^2\left(\frac{\alpha-a}{a}\right)T_p=\frac{\alpha^2}{a}(\alpha-a)T_p}$$

Comparing the two boxed results above gives the peak time.

$$\boxed{T_p=\frac{1}{\alpha-a}} \tag{21}$$

Note this result is positive (and meaningful) only if $a<\alpha$ which indicates the zero is to the right of the repeated poles. For the example system of Fig. 6, Eq. (21) predicts a peak time of

$$\boxed{T_p=\frac{1}{\alpha-a}=\frac{1}{10-4}\approx 0.17\text{ (sec)}}$$

This is consistent with the measured result shown in Fig. 6.

Substituting from Eq. (21) into Eq. (20) gives the corresponding maximum value of $x(t)$. The percent overshoot is then defined to be

$$\begin{aligned}\%OS&=100\left(x(T_p)-1\right)=100\left[-e^{-\alpha T_p}+\left(\frac{\alpha^2-a\alpha}{a}\right)T_pe^{-\alpha T_p}\right]\\&=100\left[\left(\frac{\alpha^2-a\alpha}{a}\right)T_p-1\right]e^{-\alpha T_p}\\&=100\left[\alpha\left(\frac{\cancel{\alpha-a}}{a}\right)\left(\frac{1}{\cancel{\alpha-a}}\right)-1\right]e^{-\alpha/(\alpha-a)}\\&=100\left(\frac{\alpha}{a}-1\right)e^{-\alpha/(\alpha-a)}\end{aligned}$$

Now, define the ratio $\beta\triangleq a/\alpha$ and the above expression can be rewritten as follows.

$$\%OS=100\left(\frac{\alpha}{a}-1\right)e^{-\alpha/(\alpha-a)}=100\left(\frac{1}{\beta}-1\right)e^{-\alpha/\alpha(1-a/\alpha)}$$

$$\Rightarrow\boxed{\%OS=100\left(\frac{1}{\beta}-1\right)e^{-1/(1-\beta)}} \tag{22}$$

Using the results in Eq. (22), Fig. 5 can now be modified to include the results for Case 2 critically damped systems. See Fig. 7 below. Notice the percent overshoot for critically damped systems is defined only for systems with $a < \alpha$, or equivalently, $\beta < 1$. Eq. (22) predicts a percent overshoot of 28.3% for $\beta = 0.4$ which is consistent with the measured results for the example system shown above in Fig. 6. See boxed result in Fig. 7.

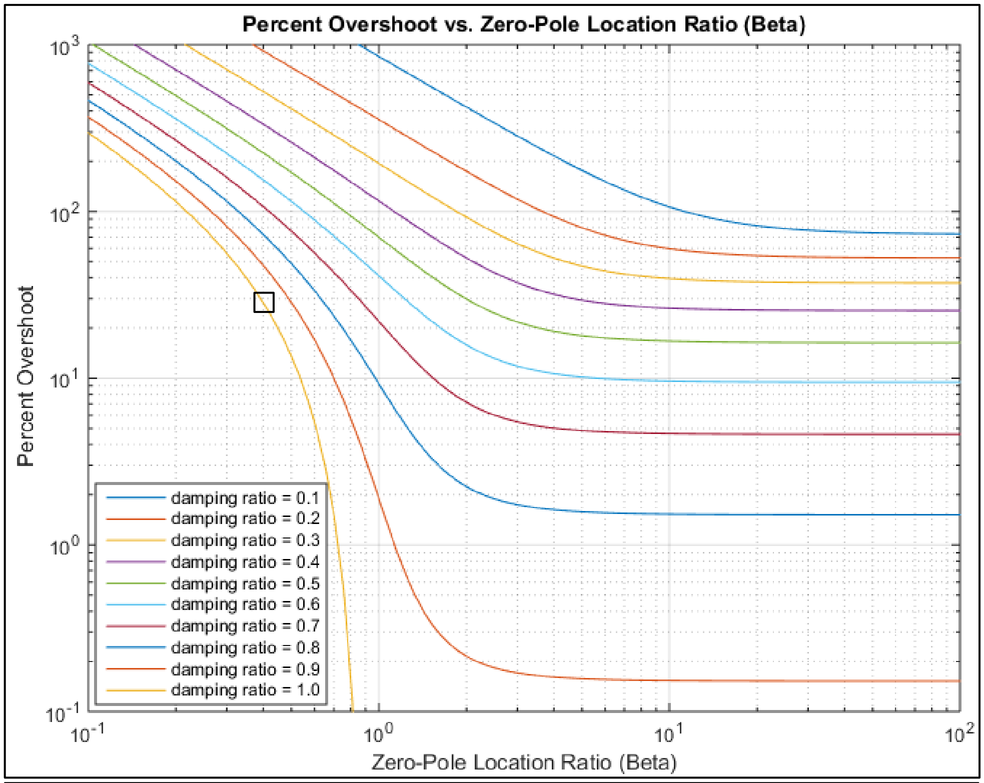

Fig. 7 Percent Overshoot vs. Zero-Pole Location Ratio β

<u>Case 2 Over-damped Systems</u>:

The general form of the ***normalized*** transfer functions of ***over damped***, ***second-order systems*** with a real zero can be written as follows. For convenience in the latter analysis, it is assumed here that $c > b$.

$$\frac{X}{R}(s) = \left(\frac{bc}{a}\right)\left[\frac{s+a}{(s+b)(s+c)}\right]$$

To analyze these systems, first separate the transfer function into two parts as done above.

$$\frac{X}{R}(s) = \left(\frac{1}{a}\right)s\left[\frac{bc}{(s+b)(s+c)}\right] + \left[\frac{bc}{(s+b)(s+c)}\right] = \left(\frac{1}{a}\right)s\left[\frac{Y}{R}(s)\right] + \left[\frac{Y}{R}(s)\right] = \left(\frac{1}{a}\right)\left[\frac{\dot{Y}}{R}(s)\right] + \left[\frac{Y}{R}(s)\right]$$

Letting $R(s) = \frac{1}{s}$ for a unit-step input and using Laplace transform tables, the response function $y(t)$ can be written as

$$y(t) = 1 - \frac{c}{(c-b)}\, e^{-bt} + \frac{b}{(c-b)}\, e^{-ct} \qquad (23)$$

Differentiating this result gives

$$\boxed{\dot{y}(t)=\frac{bc}{(c-b)}\,e^{-bt}-\frac{bc}{(c-b)}\,e^{-ct}} \tag{24}$$

Combining these results gives the response function for Case 2 over-damped systems.

$$x(t)=y(t)+\tfrac{1}{a}\,\dot{y}(t)=\left[1-\frac{c}{(c-b)}\,e^{-bt}+\frac{b}{(c-b)}\,e^{-ct}\right]+\tfrac{1}{a}\left[\frac{bc}{(c-b)}\,e^{-bt}-\frac{bc}{(c-b)}\,e^{-ct}\right]$$

$$=1+\left[\frac{bc-ac}{a(c-b)}\right]e^{-bt}+\left[\frac{ab-bc}{a(c-b)}\right]e^{-ct}$$

$$\Rightarrow \boxed{x(t)=1+\frac{c}{a}\left[\frac{b-a}{c-b}\right]e^{-bt}+\frac{b}{a}\left[\frac{a-c}{c-b}\right]e^{-ct}}$$

To find the peak time, now set the derivative of $x(t)$ to zero.

$$\dot{x}(T_p)=-\frac{bc}{a}\left[\frac{b-a}{c-b}\right]e^{-bT_p}-\frac{bc}{a}\left[\frac{a-c}{c-b}\right]e^{-cT_p}=-\frac{bc}{a(c-b)}\Big[\big(b-a\big)e^{-bT_p}+\big(a-c\big)e^{-cT_p}\Big]=0$$

Setting the term in square brackets to zero gives

$$\big(b-a\big)e^{-bT_p}=\big(c-a\big)e^{-cT_p}$$

Taking the natural log of both sides of this equation and solving for T_p gives

$$\ln\big(b-a\big)+\ln\left(e^{-bT_p}\right)=\ln\big(c-a\big)+\ln\left(e^{-cT_p}\right)$$

$$\Rightarrow \quad \ln\big(b-a\big)-bT_p=\ln\big(c-a\big)-cT_p$$

$$\Rightarrow \quad (c-b)T_p=\ln\big(c-a\big)-\ln\big(b-a\big)$$

$$\Rightarrow \boxed{T_p=\frac{\ln\big(c-a\big)-\ln\big(b-a\big)}{c-b}} \tag{25}$$

Note that Eq. (25) provides a solution for T_p only if $c>b>a$. This means the zero must be located to the right of the two real poles. The percent overshoot for these systems is defined to be

$$\boxed{\%OS=100\big(x(T_p)-1\big)=100\left[\frac{c}{a}\left(\frac{b-a}{c-b}\right)e^{-bT_p}+\frac{b}{a}\left(\frac{a-c}{c-b}\right)e^{-cT_p}\right]} \quad \big(c>b>a\big) \tag{26}$$

To check these results, consider the following system.

$$\boxed{\frac{X}{R}\big(s\big)=\left(\frac{bc}{a}\right)\left[\frac{s+a}{\big(s+b\big)\big(s+c\big)}\right]=\frac{10(s+1)}{(s+2)(s+5)}} \tag{27}$$

Using Eqs. (25) and (26) with $c=5$, $b=2$, and $a=1$, the peak time and percent overshoot of this system are

$$T_p = \frac{\ln(c-a)-\ln(b-a)}{c-b} = \frac{\ln(5-1)-\ln(2-1)}{3} \approx 0.462 \text{ (sec)} \tag{28}$$

$$\begin{aligned} \%OS &= 100\left[\frac{c}{a}\left(\frac{b-a}{c-b}\right)e^{-bT_p} + \frac{b}{a}\left(\frac{a-c}{c-b}\right)e^{-cT_p}\right] = 100\left[5\left(\frac{2-1}{5-2}\right)e^{-2(0.462)} + 2\left(\frac{1-5}{5-2}\right)e^{-5(0.462)}\right] \\ &= \frac{100}{3}\left[5e^{-0.9242} - 8e^{-2.31}\right] \quad \Rightarrow \boxed{\%OS \approx 39.7\%} \end{aligned} \tag{29}$$

Fig. 8 below shows the unit step response of three systems with transfer functions of the following form with values of $a = \{1, 3, 10\}$.

$$\frac{X}{R}(s) = \left(\frac{bc}{a}\right)\left[\frac{s+a}{(s+b)(s+c)}\right] = \frac{(10/a)(s+a)}{(s+2)(s+5)} \quad (b=2, c=5)$$

Note that for the systems with $a > b$, the system exhibits no overshoot. This is consistent with the fact that Eq. (25) for peak time has no solution if $a > b$. However, the system with $a < b$ does have overshoot. The measured peak time and percent overshoot shown for the system having $a = 1$ are consistent with those calculated above in Eqs. (28) and (29).