Introductory Motion and Control
Rationale of the Tustin Approximation

The Tustin approximation is one of the methods commonly used to transform continuous transfer functions into discrete transfer functions. Recall that discrete transfer functions can be used to generate difference equations which can be coded into digital computer programs to simulate the differential equation associated with the transfer function. This process is quite useful when continuous compensators are to be implemented on digital hardware.

Justification of the Tustin approximation is based on the numerical method used to approximate the integral and derivative terms in a proportional-integral-derivative (PID) compensator. Specifically, the trapezoidal rule is used to estimate integrals of the system error (associated with the integral term) and the derivative command signal (associated with the derivative term). Details of the numerical approximation are provided below.

As with any numerical method, this method provides an approximation of the original continuous transfer function. The accuracy of the approximation is usually application dependent. More details on this method may be found in Franklin, Powell, and Emami-Naeini, *Feedback Control of Dynamic Systems*, Prentice-Hall, 6th Ed. 2010.

<u>Integral Term</u>

The command signal $u_{\text{int}}(t)$ associated with the integral portion of the PID compensator is defined as

$$\boxed{u_{\text{int}}(t) \triangleq k_{\text{int}} \int_0^t e(\tau)\, d\tau} \tag{1}$$

Here, $e(t)$ represents the system error. Given a discrete sampling time of T, the integral in Eq. (1) can be broken into two parts as follows.

$$u_{\text{int}}(kT) = k_{\text{int}} \int_0^{kT} e(\tau)\, d\tau = \left(k_{\text{int}} \int_0^{(k-1)T} e(\tau)\, d\tau \right) + \left(k_{\text{int}} \int_{(k-1)T}^{kT} e(\tau)\, d\tau \right) \triangleq u_{\text{int}}\left((k-1)T\right) + k_{\text{int}} \int_{(k-1)T}^{kT} e(\tau)\, d\tau \tag{2}$$

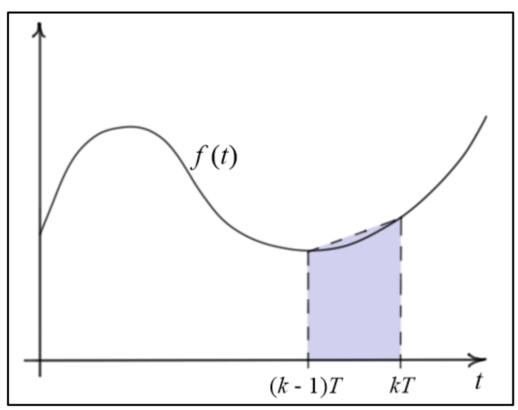
To simplify the notation, the following definitions are used.

$$\ldots,\ u_{\text{int}}(k-1) \triangleq u_{\text{int}}\left((k-1)T\right), \quad u_{\text{int}}(k) \triangleq u_{\text{int}}(kT), \quad u_{\text{int}}(k+1) \triangleq u_{\text{int}}\left((k+1)T\right), \ldots$$

$$\ldots,\ e(k-1) \triangleq e\left((k-1)T\right), \quad e(k) \triangleq e(kT), \quad e(k+1) \triangleq e\left((k+1)T\right), \ldots$$

Using these definitions, Eq. (2) can be rewritten as follows.

$$\boxed{u_{\text{int}}(k) = u_{\text{int}}(k-1) + k_{\text{int}} \int_{(k-1)T}^{kT} e(\tau)\, d\tau} \tag{3}$$



The figure to the right shows that the area under the function $f(t)$ from $(k-1)T$ to kT can be estimated using the blue trapezoidal area. Using this approach, the integral of Eq. (3) can be approximated as

$$\boxed{\int_{(k-1)T}^{kT} e(\tau)\,d\tau \approx \tfrac{1}{2}\big(e(k)+e(k-1)\big)T} \tag{4}$$

Substituting from Eq. (4) into Eq. (3) gives

$$\boxed{u_{\text{int}}(k) \approx u_{\text{int}}(k-1)+\tfrac{1}{2}k_{\text{int}}\big(e(k)+e(k-1)\big)T} \tag{5}$$

Eq. (5) can be expressed in the z domain as follows.

$$U_{\text{int}}(z) = z^{-1}U_{\text{int}}(z)+\tfrac{1}{2}k_{\text{int}}T\left[E(z)+z^{-1}E(z)\right] \tag{6}$$

Multiplying through by z and rearranging terms gives the discrete transfer function from the error signal to the integral command.

$$(z-1)U_{\text{int}}(z) = \tfrac{1}{2}k_{\text{int}}T\,(z+1)\,E(z) \quad \Rightarrow \boxed{\frac{U_{\text{int}}}{E}(z) = \tfrac{1}{2}k_{\text{int}}T\left[\frac{z+1}{z-1}\right]} \tag{7}$$

<u>Derivative Term</u>

The command signal $u_d(t)$ associated with the derivative portion of the PID compensator is defined as

$$\boxed{u_d(t) \triangleq k_d\,\dot{e}(t)} \tag{8}$$

To apply the trapezoidal rule, first integrate Eq. (8).

$$\int_{(k-1)T}^{kT} u_d(\tau)\,d\tau = \int_{e(k-1)}^{e(k)} k_d\,d\varepsilon = k_d\big(e(k)-e(k-1)\big)$$

The integral on the left side of the equation is now approximated using the trapezoidal rule giving

$$\boxed{\tfrac{1}{2}T\big(u_d(k)+u_d(k-1)\big) = k_d\big(e(k)-e(k-1)\big)} \tag{9}$$

Eq. (9) can be expressed in the z domain as follows.

$$\tfrac{1}{2}T\big(U_d(z)+z^{-1}U_d(z)\big) = k_d\big(E(z)-z^{-1}E(z)\big)$$

Multiplying through by z and rearranging terms gives the discrete transfer function from the error signal to the derivative command.

$$\tfrac{1}{2}T\,(z+1)U_d(z) = k_d\,(z-1)\,E(z) \quad \Rightarrow \boxed{\frac{U_d}{E}(z) = \frac{2k_d}{T}\left[\frac{z-1}{z+1}\right]} \tag{10}$$

<u>Tustin Approximation</u>

The discrete and continuous transfer functions for the integral and derivative terms of PID compensators are compared in Eqs. (11) and (12) below.

Continuous integral form: $\boxed{\dfrac{U_{\text{int}}}{E}(s) = \dfrac{k_{\text{int}}}{s}}$ Discrete integral form: $\boxed{\dfrac{U_{\text{int}}}{E}(z) = \tfrac{1}{2}k_{\text{int}}T\left[\dfrac{z+1}{z-1}\right]}$ (11)

Continuous derivative form: $\boxed{\dfrac{U_d}{E}(s) = k_d\,s}$ Discrete derivative form: $\boxed{\dfrac{U_d}{E}(z) = \dfrac{2k_d}{T}\left[\dfrac{z-1}{z+1}\right]}$ (12)

Note in both cases that the s and z portions of the derivative and integral transfer functions are algebraic inverses of each other – the continuous transfer functions have s and $\frac{1}{s}$, and the discrete transfer functions have $\frac{2}{T}\left[\frac{z-1}{z+1}\right]$ and $\frac{T}{2}\left[\frac{z+1}{z-1}\right]$. Furthermore, it is clear from observing Eqs. (11) and (12) that the discrete transfer functions can be found by simply replacing the variable s in the continuous transfer functions with $\frac{2}{T}\left[\frac{z-1}{z+1}\right]$.

Replacing s in a continuous transfer function with $\frac{2}{T}\left[\frac{z-1}{z+1}\right]$ to create an approximate discrete transfer function is referred to as Tustin's approximation. It is one method of converting continuous transfer functions into approximate discrete counterparts. Another common method is the matched pole-zero (MPZ) method. The MPZ method is based on mapping the poles and zeros of a continuous transfer function using the relationship $z = e^{sT}$ and preserving the low frequency gain. As mentioned above, the accuracy of any numerical approximation is usually application dependent.