Introductory Control Systems
Exercises #16 – State-Space Equations and Transfer Functions

1. Speed control of a car in the presence of a disturbance force is described by the boxed equations. The desired speed is $r(t)$, the actual speed is $v(t)$, the speed error is $e(t)$, the driving force on the car is $f_a(t)$, the disturbance force on the car is $f_d(t)$, and the net force on the car is $f_{net}(t)$.

$$\frac{dv}{dt}+3v = f_{net}(t)$$
$$f_{net}(t) = f_a(t) - f_d(t)$$
$$f_a(t) = Ke(t)$$
$$e(t) = r(t) - v(t)$$

a) Express the equations in state-space form with output variables $v(t)$ and $e(t)$. b) Using the state-space equations, find the transfer functions $\frac{V}{R}(s)$, $\frac{E}{R}(s)$, $\frac{V}{F_d}(s)$, and $\frac{E}{F_d}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{V}{R}(s)$.

Answers:

a) $\dot{x}_1 = \left[-(K+3)\right]x_1 + \left[K \quad -1\right]\begin{Bmatrix} r \\ f_d \end{Bmatrix}$ $\quad$ $\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

b) $\frac{V}{R}(s) = \frac{K}{s+(K+3)}$ $\quad$ $\frac{V}{F_d}(s) = \frac{-1}{s+(K+3)}$ $\quad$ $\frac{E}{R}(s) = \frac{s+3}{s+(K+3)}$ $\quad$ $\frac{E}{F_d}(s) = \frac{1}{s+(K+3)}$

c) $\dot{x}_1 = \left[-(K+3)\right]x_1 + \left[1\right]r(t)$ $\quad$ $v(t) = \left[K\right]x_1(t)$

2. The boxed equations describe ***position control*** of a spring-mass system using ***proportional*** (P) control. The ***desired position*** of the mass is $r(t)$, the ***actual position*** is $y(t)$, the ***position error*** is $e(t)$, and the ***actuator force*** applied to the mass is $f(t)$.

$$e(t) = r(t) - y(t)$$
$$f(t) = K\,e(t)$$
$$\ddot{y} + 4y = f(t)$$

a) Express the equations in state-space form with output variables y, $\dot{y}$, and e. b) Using the state-space equations, find the transfer functions $\frac{Y}{R}(s)$ and $\frac{E}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{E}{R}(s)$.

Answers:

a) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(K+4) & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ K \end{Bmatrix} u$ $\quad$ $\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ e \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} u$

b) $\frac{Y}{R}(s) = \frac{K}{s^2+K+4}$ $\quad$ $\frac{E}{R}(s) = \frac{s^2+4}{s^2+K+4}$

c) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(K+4) & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$ $\quad$ $e = \left[-K \quad 0\right]\begin{Bmatrix} x_1 \\ x_1 \end{Bmatrix} + \left[1\right]u$

3. The boxed equations describe position control of a spring-mass-damper system using proportional-derivative control. The desired position of the mass is $r(t)$, the actual position is $x(t)$, the position error is $e(t)$, and the actuating force is $f(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t) - x(t) \\ f(t) &= \dot{e}(t) + 5e(t) \\ \ddot{x} + 6\dot{x} + 20x &= f(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables x and e. b) Using the state-space equations, find the transfer functions $\frac{X}{R}(s)$ and $\frac{E}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{X}{R}(s)$.

Answers:

a) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ $\quad$ $\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x \\ e \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

b) $\frac{Z_1}{U_1}(s) = \frac{X}{R}(s) = \frac{s+5}{s^2+7s+25}$ $\quad$ $\frac{Z_2}{U_1}(s) = \frac{E}{R}(s) = \frac{s^2+6s+20}{s^2+7s+25}$

c) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} r(t)$ $\quad$ $x(t) = \begin{bmatrix} 5 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

4. Position control of a hydraulic actuator using proportional-integral control is described by the boxed equations. The desired position is $r(t)$, the actual position is $x(t)$, the position error is $e(t)$, and the actuating forces is $f(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t) - x(t) \\ f(t) &= e(t) + 3\int_0^t e(t)dt \\ \ddot{x} + 5\dot{x} &= 2f(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables x and e. b) Using the state-space equations, find the transfer functions $\frac{X}{R}(s)$ and $\frac{E}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{X}{R}(s)$.

Answers:

a) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -2 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 6 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ $\quad$ $\begin{Bmatrix} x \\ e \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

b) $\frac{X}{R}(s) = \frac{2(s+3)}{s^3+5s^2+2s+6}$ $\quad$ $\frac{E}{R}(s) = \frac{s^2(s+5)}{s^3+5s^2+2s+6}$

c) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -2 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} r(t)$ $\quad$ $x(t) = 6x_1 + 2x_2 = \begin{bmatrix} 6 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

5. The boxed equations describe speed control of a rotating mass system using proportional control. The desired rotational speed is $r(t)$, the actual rotational speed is $\omega(t)$, the rotational speed error is $e(t)$, the input voltage to the motor drive actuator is $v(t)$, and the actuator torque applied to the mass is $M(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t)-\omega(t) \\ v(t) &= K e(t) \\ \dot{M}+8M &= v(t) \\ \dot{\omega}+7\omega &= 3M(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables ω and M. b) Using the state-space equations, find the transfer functions $\frac{\omega}{R}(s)$ and $\frac{M}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{M}{R}(s)$.

Answers:

a) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -7 & 3 \\ -K & -8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ K \end{Bmatrix} u$ $\quad$ $\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} u$

b) $\frac{\omega}{R}(s) = \frac{3K}{s^2+15s+(56+3K)}$ $\quad$ $\frac{M}{R}(s) = \frac{K(s+7)}{s^2+15s+(56+3K)}$

c) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(56+3K) & -15 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} r$ $\quad$ $M = \begin{bmatrix} 7K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

6. The boxed equations describe ***position control*** of a spring-mass system using proportional control with a linear actuator. The ***desired position*** of the mass is $r(t)$, the ***actual position*** is $y(t)$, the ***position error*** is $e(t)$, and the ***actuator force*** applied to the mass is $f(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t)-y(t) \\ v(t) &= 10\, e(t) \\ \dot{f}+3f &= v(t) \\ \ddot{y}+9y &= f(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables y and v. b) Using the state-space equations, find the transfer functions $\frac{Y}{R}(s)$ and $\frac{V}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{Y}{R}(s)$.

Answers:

a) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -10 & -3 & 0 \\ -9 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 10 \\ 0 \end{Bmatrix} u$ $\quad$ $\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -10 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} u$

b) $\frac{Y}{R}(s) = \frac{10}{s^3+3s^2+9s+37}$ $\quad$ $\frac{V}{R}(s) = \frac{10\left(s^3+3s^2+9s+27\right)}{s^3+3s^2+9s+37}$

c) $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37 & -9 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} r$ $\quad$ $y = \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$