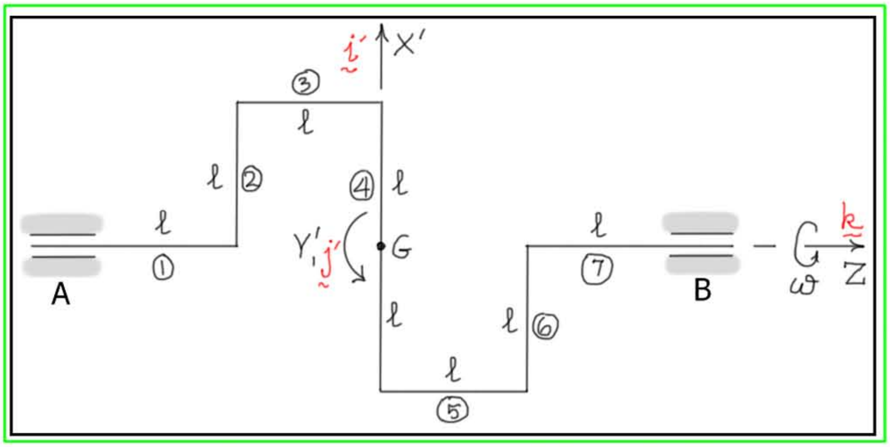
Example #11 – Intermediate Dynamics: Kinetic Energy

The figure shows a ***simple crank shaft*** consisting of ***seven segments***, each considered to be a ***slender bar***. Each segment of ***length*** ℓ has ***mass*** m. There are six segments of length ℓ and one segment of length 2ℓ (segment 4). The mass center of the system is G and is located on the axis of rotation.



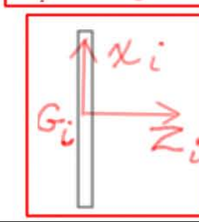
Reference frames:

$R: \quad \underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ (fixed frame) $\qquad S: \quad \underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k}$ (rotates with the shaft)

Find:

$\underset{\sim}{H}_G$ … angular momentum of the system about its mass center, G

K … kinetic energy of the system



Solution: (components in rotating shaft frame)

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{i}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{j}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{k} \end{Bmatrix} = \begin{bmatrix} I^G_{X'X'} & -I^G_{X'Y'} & -I^G_{X'Z} \\ -I^G_{Y'X'} & I^G_{Y'Y'} & -I^G_{Y'Z} \\ -I^G_{ZX'} & -I^G_{ZY'} & I^G_{ZZ} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} = \begin{Bmatrix} -I^G_{X'Z}\omega \\ -I^G_{Y'Z}\omega \\ I^G_{ZZ}\omega \end{Bmatrix}$$

$$I^A_{ii} = I^G_{ii} + m\, d_i^2 \quad (i = x, y, \text{ or } z)$$
$$I^A_{ij} = I^G_{ij} + m\, c_i\, c_j \quad (i \text{ and } j = x, y, \text{ or } z)$$

(parallel axes theorems)

Using the parallel axes theorems for moments and products of inertia:

$$I^G_{ZZ} = \sum_{i=1}^{7} \left(I^G_{ZZ}\right)_i = 0 + \tfrac{1}{3}m\ell^2 + m\ell^2 + \tfrac{1}{12}(2m)(2\ell)^2 + m\ell^2 + \tfrac{1}{3}m\ell^2 + 0 = \boxed{\tfrac{10}{3}m\ell^2}$$

$$I^G_{X'Z} = \sum_{i=1}^{7} \left(I^G_{X'Z}\right)_i = 0 + m(\tfrac{\ell}{2})(-\ell) + m(\ell)(-\tfrac{\ell}{2}) + 0 + m(-\ell)(\tfrac{\ell}{2}) + m(-\tfrac{\ell}{2})(\ell) + 0 = \boxed{-2m\ell^2}$$

$I^G_{Y'Z} \equiv 0$ (since the $X'Z$ plane is a ***plane*** of ***symmetry***, so ***products*** associated with Y' are ***zero***)

So,

$$\boxed{\underset{\sim}{H}_G = 2m\ell^2\omega\, \underset{\sim}{i}' + (\tfrac{10}{3})m\ell^2\omega\, \underset{\sim}{k}}$$

The ***kinetic energy*** of the crank shaft is found from the ***velocity*** and ***angular momentum*** vectors to be

$$K = \underbrace{\tfrac{1}{2}m({}^R\underset{\sim}{v}_G)^2}_{\text{zero}} + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2}(\omega \underset{\sim}{k}) \cdot \underset{\sim}{H}_G = \boxed{\tfrac{10}{6}m\ell^2\omega^2}$$