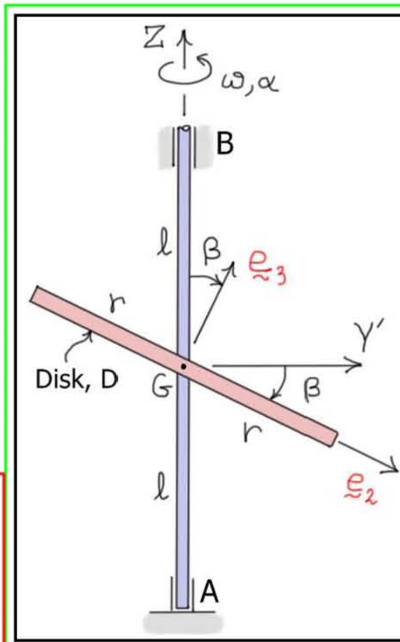
Example 12 – Intermediate Dynamics: Angular Momentum and Kinetic Energy



Reference frames: (R is the fixed frame)

$S: \; \underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k}$ (rotates with the shaft; aligned with the shaft)

$D: \underset{\sim}{i}', \underset{\sim}{e}_2, \underset{\sim}{e}_3$ (rotates with the shaft; aligned with the disk)

Find:

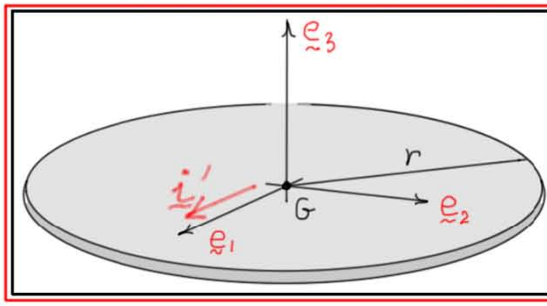
$\underset{\sim}{H}_G$ … ***angular momentum*** of the disk about its mass center, G

K … ***kinetic energy*** of the disk

Solution:

$$ {}^R\underset{\sim}{\omega}_D = \omega\, \underset{\sim}{k} = \omega\left(-S_\beta\, \underset{\sim}{e}_2 + C_\beta\, \underset{\sim}{e}_3\right) $$

$$ \left[I_G\right]_D = m r^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} $$



$$ \begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{i}' \\ \underset{\sim}{H}_G \cdot \underset{\sim}{e}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{e}_3 \end{Bmatrix} = m r^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ -\omega S_\beta \\ \omega C_\beta \end{Bmatrix} = m r^2 \begin{Bmatrix} 0 \\ -\frac{1}{4}\omega S_\beta \\ \frac{1}{2}\omega C_\beta \end{Bmatrix} \Rightarrow \boxed{\underset{\sim}{H}_G = \tfrac{1}{4} m r^2 \omega\left(-S_\beta\, \underset{\sim}{e}_2 + 2C_\beta\, \underset{\sim}{e}_3\right)} $$

Expressing $\underset{\sim}{H}_G$ in the reference frame S:

$$ \underset{\sim}{H}_G = \tfrac{1}{4} m r^2 \omega\left(-S_\beta\, \underset{\sim}{e}_2 + 2C_\beta\, \underset{\sim}{e}_3\right) = \tfrac{1}{4} m r^2 \omega\left[-S_\beta\left(C_\beta\, \underset{\sim}{j}' - S_\beta\, \underset{\sim}{k}\right) + 2C_\beta\left(S_\beta\, \underset{\sim}{j}' + C_\beta\, \underset{\sim}{k}\right)\right] $$

Or,

$$ \boxed{\underset{\sim}{H}_G = \tfrac{1}{4} m r^2 \omega\left[\left(S_\beta C_\beta\right) \underset{\sim}{j}' + \left(2C_\beta^2 + S_\beta^2\right) \underset{\sim}{k}\right] = \tfrac{1}{4} m r^2 \omega\left[\left(S_\beta C_\beta\right) \underset{\sim}{j}' + \left(C_\beta^2 + 1\right) \underset{\sim}{k}\right]} $$

We also know that $\boxed{\underset{\sim}{H}_G = \left(-I_{X'Z}\,\omega\right) \underset{\sim}{i}' + \left(-I_{Y'Z}\,\omega\right) \underset{\sim}{j}' + \left(I_{ZZ}\,\omega\right) \underset{\sim}{k}}$.

Comparing these two results, we find: $\boxed{I_{X'Z} = 0}$, $\boxed{I_{Y'Z} = -\tfrac{1}{4} m r^2 S_\beta C_\beta}$, $\boxed{I_{ZZ} = \tfrac{1}{4} m r^2 \left(C_\beta^2 + 1\right)}$

The ***kinetic energy*** of the disk is found from the ***velocity*** and ***angular momentum*** vectors:

$$ K = \underbrace{\tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G\right)^2}_{\text{zero}} + \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_D \cdot \underset{\sim}{H}_G = \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_D \cdot \underset{\sim}{H}_G = \tfrac{1}{2}(\omega \underset{\sim}{k}) \cdot \underset{\sim}{H}_G = \boxed{\tfrac{1}{8} m r^2 \left(C_\beta^2 + 1\right) \omega^2} $$