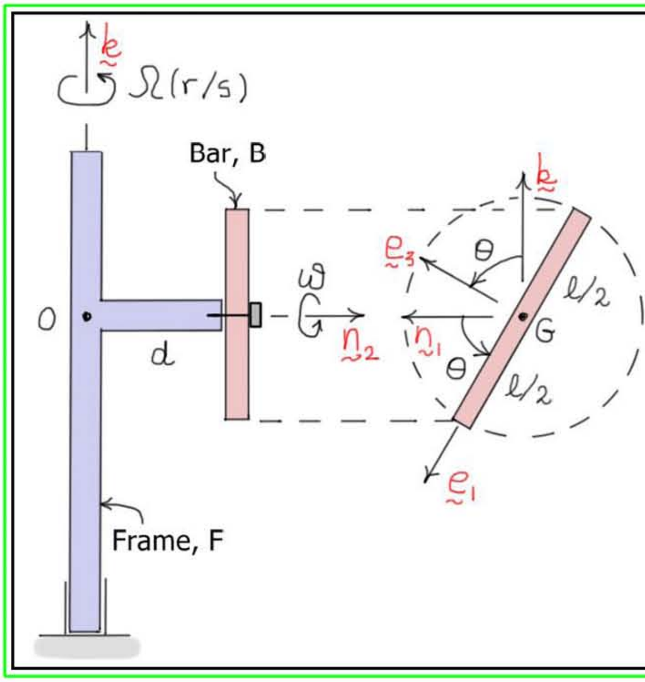
Example #13 – Intermediate Dynamics: Angular Momentum and Kinetic Energy

Reference frames: (R is the fixed frame)

$F:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{k})$ … (rotates with frame F)

$B:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{e}_3)$ … (rotates with the bar B)



Find:

$\underset{\sim}{H}_G$ … ***angular momentum*** of B about its mass-center, G

K … ***kinetic energy*** of B

Solution:

The ***angular velocity vector*** and the ***inertia matrix*** about ***body-fixed axes*** are:

$$[I_G]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m\ell^2 & 0 \\ 0 & 0 & \frac{1}{12}m\ell^2 \end{bmatrix}$$

and $\quad {}^R\underset{\sim}{\omega}_B = \omega\,\underset{\sim}{n}_2 + \Omega\,\underset{\sim}{k} = \omega\,\underset{\sim}{n}_2 + \Omega(-S_\theta \underset{\sim}{e}_1 + C_\theta \underset{\sim}{e}_3)$

$$\Rightarrow \quad {}^R\underset{\sim}{\omega}_B = (-\Omega S_\theta)\underset{\sim}{e}_1 + \omega\,\underset{\sim}{n}_2 + (\Omega C_\theta)\underset{\sim}{e}_3$$

The ***body-fixed components*** of the ***angular momentum*** vector are

$$\begin{Bmatrix} \underset{\sim}{H}_G \cdot \underset{\sim}{e}_1 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{n}_2 \\ \underset{\sim}{H}_G \cdot \underset{\sim}{e}_3 \end{Bmatrix} = \frac{m\ell^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -\Omega S_\theta \\ \omega \\ \Omega C_\theta \end{Bmatrix} = \frac{m\ell^2}{12}\begin{Bmatrix} 0 \\ \omega \\ \Omega C_\theta \end{Bmatrix} = \frac{m\ell^2}{12}\left(\omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right)$$

The ***angular momentum*** can also be expressed using unit vectors fixed in F:

$$\underset{\sim}{H}_G = \tfrac{m\ell^2}{12}(\omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3) = \tfrac{m\ell^2}{12}\left[\omega\,\underset{\sim}{n}_2 + \Omega C_\theta(S_\theta \underset{\sim}{n}_1 + C_\theta \underset{\sim}{k})\right] = \tfrac{m\ell^2}{12}\left[\Omega C_\theta S_\theta \underset{\sim}{n}_1 + \omega\,\underset{\sim}{n}_2 + \Omega C_\theta^2 \underset{\sim}{k}\right]$$

The kinetic energy of the bar is calculated using the angular momentum vector as follows.

$$K = K_{translation} + K_{rotation} = \tfrac{1}{2}m({}^R\underset{\sim}{v}_G)^2 + \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G$$

$$= \tfrac{1}{2}md^2\Omega^2 + \tfrac{1}{2}\left(-\Omega S_\theta\,\underset{\sim}{e}_1 + \omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right)\cdot \tfrac{m\ell^2}{12}\left(\omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right)$$

$$\Rightarrow \quad K = \tfrac{1}{2}md^2\Omega^2 + \tfrac{m\ell^2}{24}\left(\omega^2 + \Omega^2 C_\theta^2\right)$$