

Example #20 – Intermediate Dynamics: Lagrange's Equations (1 DOF system)

Given:

$$m_B = m_D = m$$

D rolls without slipping on the circular surface

Active Forces/Torques:

Torque M and weight forces

Find:

Differential equation of motion of the system

Solution: (using θ as the generalized coordinate)

Using θ as the single *generalized coordinate*, the *equation of motion* of the system can be found using Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = F_\theta \quad \text{with} \quad L = K - V = K_B + K_D - V_B - V_D \quad (\text{Lagrangian})$$

$$K_D = \frac{1}{2} \omega_D \cdot H_C = \frac{1}{2} I_C \dot{\phi}^2 = \frac{1}{2} \left(\frac{1}{2} m r^2 + m r^2 \right) \dot{\phi}^2 = \frac{3}{4} m r^2 \dot{\phi}^2 \quad (\text{fixed axis rotation})$$

$$K_B = \frac{1}{2} \omega_B \cdot H_O = \frac{1}{2} I_O \dot{\theta}^2 = \frac{1}{2} \left(\frac{1}{3} m l^2 \right) \dot{\theta}^2 = \frac{1}{6} m l^2 \dot{\theta}^2 \quad (\text{fixed axis rotation})$$

$$V = V_D + V_B = -m g l C_\theta - \frac{1}{2} m g l C_\theta = -\frac{3}{2} m g l C_\theta \quad (\text{datum as shown in diagram})$$

Before using Lagrange's equation, angle ϕ must be eliminated from the equations. Using the concept of instantaneous centers of zero velocity, note that $v_P = l \dot{\theta} = -r \dot{\phi}$, or $\dot{\phi} = -(\ell / r) \dot{\theta}$.

$$L = \frac{3}{4} m \cancel{\nu}^2 \left(\ell / \cancel{\nu} \right)^2 \dot{\theta}^2 + \frac{1}{6} m l^2 \dot{\theta}^2 + \frac{3}{2} m g l C_\theta = \frac{11}{12} m l^2 \dot{\theta}^2 + \frac{3}{2} m g l C_\theta$$

Generalized active force F_θ and the derivatives of the Lagrangian:

$$F_\theta = M \underline{k} \cdot \frac{\partial}{\partial \dot{\theta}} (\omega_B) = M \underline{k} \cdot \underline{k} = M(t)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{11}{6} m l^2 \dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{11}{6} m l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{3}{2} m g l S_\theta$$

Substituting into Lagrange's equation gives the equation of motion

$$\frac{11}{6} m l^2 \ddot{\theta} + \frac{3}{2} m g l S_\theta = M(t) \quad (\text{non-linear, second-order, ordinary, differential equation})$$

