Example #21 – Intermediate Dynamics: Lagrange's Equations (2 DOF system)

Given:

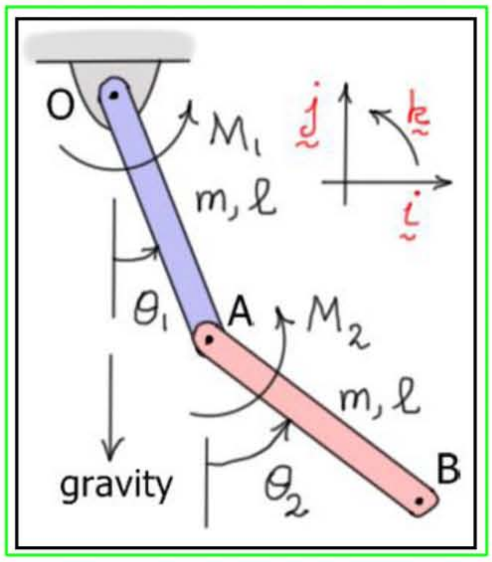
driving torques $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$

Active Forces/Torques:

$\underset{\sim}{M}_1$, $\underset{\sim}{M}_2$, and weight forces

Find:

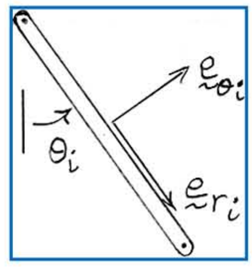
differential equations of motion of the system



Solution: (using θ_1 and θ_2 as the generalized coordinates)

Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=F_{\theta_1} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=F_{\theta_2}$$



Typical Link

Kinematics:

$$\underset{\sim}{v}_{G_1}=\underset{\sim}{v}_O+\underset{\sim}{v}_{G_1/O}=\tfrac{1}{2}\ell\dot{\theta}_1\underset{\sim}{e}_{\theta_1} \quad \Rightarrow \quad v_{G_1}^2=\underset{\sim}{v}_{G_1}\cdot\underset{\sim}{v}_{G_1}=\tfrac{1}{4}\ell^2\dot{\theta}_1^2$$

$$\underset{\sim}{v}_{G_2}=\underset{\sim}{v}_A+\underset{\sim}{v}_{G_2/A}=\underset{\sim}{v}_O+\underset{\sim}{v}_{A/O}+\underset{\sim}{v}_{G_2/A}=\ell\dot{\theta}_1\underset{\sim}{e}_{\theta_1}+\tfrac{1}{2}\ell\dot{\theta}_2\underset{\sim}{e}_{\theta_2}$$

$$\Rightarrow \quad v_{G_2}^2=\underset{\sim}{v}_{G_2}\cdot\underset{\sim}{v}_{G_2}=\ell^2\dot{\theta}_1^2+\tfrac{1}{4}\ell^2\dot{\theta}_2^2+2\left(\tfrac{1}{2}\ell^2\dot{\theta}_1\dot{\theta}_2\right)\left(\underset{\sim}{e}_{\theta_1}\cdot\underset{\sim}{e}_{\theta_2}\right)=\ell^2\dot{\theta}_1^2+\tfrac{1}{4}\ell^2\dot{\theta}_2^2+\ell^2\dot{\theta}_1\dot{\theta}_2C_{2-1}$$

Lagrangian: $L=K-V=K_1+K_2-V_1-V_2$

$$K_1=\tfrac{1}{2}I_O\dot{\theta}_1^2=\tfrac{1}{2}\left(\tfrac{1}{3}m\ell^2\right)\dot{\theta}_1^2=\tfrac{1}{6}m\ell^2\dot{\theta}_1^2 \quad \text{(fixed axis rotation)}$$

$$K_2=\tfrac{1}{2}mv_{G_2}^2+\tfrac{1}{2}I_{G_2}\dot{\theta}_2^2=\tfrac{1}{2}m\ell^2\dot{\theta}_1^2+\tfrac{1}{8}m\ell^2\dot{\theta}_2^2+\tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2C_{2-1}+\tfrac{1}{24}m\ell^2\dot{\theta}_2^2 \quad \text{(general plane motion)}$$

$$=\tfrac{1}{2}m\ell^2\dot{\theta}_1^2+\tfrac{1}{6}m\ell^2\dot{\theta}_2^2+\tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2C_{2-1}$$

$$V=V_1+V_2=-\tfrac{1}{2}mg\ell C_1-mg\left(\ell C_1+\tfrac{1}{2}\ell C_2\right)=-\tfrac{3}{2}mg\ell C_1-\tfrac{1}{2}mg\ell C_2 \quad \text{(datum at O)}$$

$$L=\tfrac{2}{3}m\ell^2\dot{\theta}_1^2+\tfrac{1}{6}m\ell^2\dot{\theta}_2^2+\tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2C_{2-1}+\tfrac{3}{2}mg\ell C_1+\tfrac{1}{2}mg\ell C_2$$

Generalized Forces and Derivatives of Lagrangian:

$$F_{\theta_1}=\left(M_1\underset{\sim}{k}\cdot\frac{\partial\underset{\sim}{\omega}_1}{\partial\dot{\theta}_1}\right)+\left(-M_2\underset{\sim}{k}\cdot\frac{\partial\underset{\sim}{\omega}_1}{\partial\dot{\theta}_1}\right)+\left(M_2\underset{\sim}{k}\cdot\frac{\partial\underset{\sim}{\omega}_2}{\partial\dot{\theta}_1}\right)=M_1-M_2$$

$$F_{\theta_2}=\left(M_1\underset{\sim}{k}\cdot\frac{\partial\underset{\sim}{\omega}_1}{\partial\dot{\theta}_2}\right)+\left(-M_2\underset{\sim}{k}\cdot\frac{\partial\underset{\sim}{\omega}_1}{\partial\dot{\theta}_2}\right)+\left(M_2\underset{\sim}{k}\cdot\frac{\partial\underset{\sim}{\omega}_2}{\partial\dot{\theta}_2}\right)=M_2$$

$$\frac{\partial L}{\partial\dot{\theta}_1}=\tfrac{4}{3}m\ell^2\dot{\theta}_1+\tfrac{1}{2}m\ell^2\dot{\theta}_2C_{2-1} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)=\tfrac{4}{3}m\ell^2\ddot{\theta}_1+\tfrac{1}{2}m\ell^2C_{2-1}\ddot{\theta}_2-\tfrac{1}{2}m\ell^2\dot{\theta}_2\left(\dot{\theta}_2-\dot{\theta}_1\right)S_{2-1}$$

$$\frac{\partial L}{\partial\dot{\theta}_2}=\tfrac{1}{2}m\ell^2C_{2-1}\dot{\theta}_1+\tfrac{1}{3}m\ell^2\dot{\theta}_2 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)=\tfrac{1}{2}m\ell^2C_{2-1}\ddot{\theta}_1+\tfrac{1}{3}m\ell^2\ddot{\theta}_2-\tfrac{1}{2}m\ell^2\dot{\theta}_1\left(\dot{\theta}_2-\dot{\theta}_1\right)S_{2-1}$$

$$\frac{\partial L}{\partial\theta_1}=\tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2S_{2-1}-\tfrac{3}{2}mg\ell S_1 \qquad \frac{\partial L}{\partial\theta_2}=-\tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2S_{2-1}-\tfrac{1}{2}mg\ell S_2$$

Substituting into Lagrange's equations gives a ***coupled*** set of ***nonlinear, second-order, ordinary differential equations of motion***

$$\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1+\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_2-\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_2^2+\tfrac{3}{2}mg\ell S_1=M_1(t)-M_2(t)$$

$$\left(\tfrac{1}{2}m\ell^2C_{2-1}\right)\ddot{\theta}_1+\left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2+\left(\tfrac{1}{2}m\ell^2S_{2-1}\right)\dot{\theta}_1^2+\tfrac{1}{2}mg\ell S_2=M_2(t)$$