

Example #23 – Intermediate Dynamics: (Lagrange’s Equations – 3D, 2 DOF system)

Reference frames: ( $R$  is the fixed frame)

$F : (\underline{n}_1, \underline{n}_2, \underline{k}) \dots$  (rotates with frame  $F$ )

$B : (\underline{e}_1, \underline{n}_2, \underline{e}_3) \dots$  (rotates with the bar  $B$ )

Given:

$M_\phi(t) \dots$  applied to  $F$  by the ground

$M_\theta(t) \dots$  applied to  $B$  by  $F$

$B$  has mass  $m$ ;  $F$  is light

Find:

o **differential equations of motion** of the system

Solution: (using  $\phi$  and  $\theta$  as **generalized coordinates**)

Lagrange’s equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = F_\phi \quad \text{and} \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = F_\theta$$

Here,

$$L = K - V = K = \frac{1}{2} m d^2 \dot{\phi}^2 + \frac{m \ell^2}{24} (\dot{\theta}^2 + \dot{\phi}^2 C_\theta^2) \quad (\text{no potential energy term})$$

Generalized Forces: (associated with the driving torques)

$$F_\phi = \left( M_\theta \underline{n}_2 \cdot \frac{\partial {}^R \underline{\omega}_B}{\partial \dot{\phi}} \right) + \left( -M_\theta \underline{n}_2 \cdot \frac{\partial {}^R \underline{\omega}_F}{\partial \dot{\phi}} \right) + \left( M_\phi \underline{k} \cdot \frac{\partial {}^R \underline{\omega}_F}{\partial \dot{\phi}} \right) = \boxed{M_\phi}$$

$$F_\theta = \left( M_\theta \underline{n}_2 \cdot \frac{\partial {}^R \underline{\omega}_B}{\partial \dot{\theta}} \right) + \left( -M_\theta \underline{n}_2 \cdot \frac{\partial {}^R \underline{\omega}_F}{\partial \dot{\theta}} \right) + \left( M_\theta \underline{k} \cdot \frac{\partial {}^R \underline{\omega}_F}{\partial \dot{\theta}} \right) = \boxed{M_\theta}$$

Derivatives of Lagrangian:

$$\frac{\partial L}{\partial \dot{\phi}} = m d^2 \dot{\phi} + \frac{1}{12} m \ell^2 \dot{\phi} C_\theta^2 \quad \Rightarrow \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = (m d^2 + \frac{1}{12} m \ell^2 C_\theta^2) \ddot{\phi} - \frac{1}{6} m \ell^2 \dot{\theta} \dot{\phi} S_\theta C_\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{12} m \ell^2 \dot{\theta} \quad \Rightarrow \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{1}{12} m \ell^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \phi} = 0$$

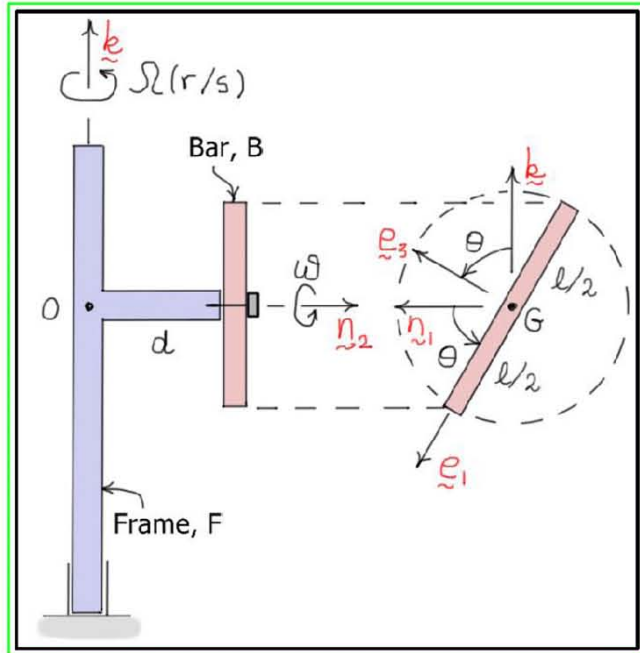
$$\frac{\partial L}{\partial \theta} = -\frac{1}{12} m \ell^2 \dot{\phi}^2 S_\theta C_\theta$$

$$\frac{d}{dt} (C_\theta^2) = 2 C_\theta (-S_\theta) \dot{\theta} = -2 S_\theta C_\theta \dot{\theta}$$

Equations of Motion:

Substituting into Lagrange’s equations gives a **coupled** set of **nonlinear, second-order, ordinary differential equations of motion**

$$\begin{aligned} (m d^2 + \frac{1}{12} m \ell^2 C_\theta^2) \ddot{\phi} - (\frac{1}{6} m \ell^2 S_\theta C_\theta) \dot{\theta} \dot{\phi} &= M_\phi(t) \\ (\frac{1}{12} m \ell^2) \ddot{\theta} + (\frac{1}{12} m \ell^2 S_\theta C_\theta) \dot{\phi}^2 &= M_\theta(t) \end{aligned}$$



Previous Results:

$${}^R \underline{\omega}_F = \dot{\phi} \underline{k} \quad {}^R \underline{\omega}_B = \dot{\theta} \underline{n}_2 + \dot{\phi} \underline{k}$$

$$K = \frac{1}{2} m d^2 \dot{\phi}^2 + \frac{m \ell^2}{24} (\dot{\theta}^2 + \dot{\phi}^2 C_\theta^2)$$