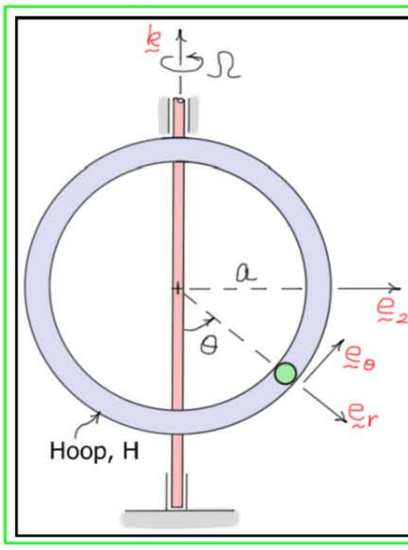
Example #24 – Intermediate Dynamics: (Linearization of Equations of Motion)



Given:

Bead B slides within the hoop H while H rotates about a vertical axis at a ***constant rate*** of Ω (r/s). The ***damped*** motion of B is described by the ***differential equation of motion***

$$\ddot{\theta} + (c/m)\dot{\theta} - \Omega^2 \sin(\theta)\cos(\theta) + (g/a)\sin(\theta) = 0$$

Find:

- ***equilibrium positions*** of B
- ***approximate linear*** equation of motion about one of those positions.

Solution:

Equilibrium positions of B can be found from the differential equation by setting $\dot{\theta} = \ddot{\theta} = 0$.

$$\Rightarrow \tfrac{g}{a}\sin(\theta) - \Omega^2 \sin(\theta)\cos(\theta) = 0 \quad \Rightarrow \quad \left(\tfrac{g}{a} - \Omega^2 \cos(\theta)\right)\sin(\theta) = 0$$

This result is true under ***two conditions***:

$$\sin(\theta) = 0 \quad \Rightarrow \quad \theta = \begin{cases} 0 \\ \pi \end{cases} \qquad \text{and} \qquad \tfrac{g}{a} - \Omega^2 \cos(\theta) = 0 \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{g}{a\,\Omega^2}\right)$$

Linearization about $\theta = 0$: (the ***first two terms*** in the equation of motion are ***linear***)

third term: $f(\theta) = \sin(\theta)\cos(\theta) \quad \Rightarrow \quad f'(\theta)\big|_{\theta=0} = \left(\cos^2(\theta) - \sin^2(\theta)\right)_{\theta=0} = 1$

$\Rightarrow$ linear approximation: $\Delta f = \left(f'(\theta)\big|_{\theta=0}\right)\Delta\theta = \Delta\theta$

fourth term: $f(\theta) = \sin(\theta) \quad \Rightarrow \quad f'(\theta)\big|_{\theta=0} = \left(\cos(\theta)\right)_{\theta=0} = 1$

$\Rightarrow$ linear approximation: $\Delta f = \left(f'(\theta)\big|_{\theta=0}\right)\Delta\theta = \Delta\theta$

Approximate linear equation of motion: $\Delta\ddot{\theta} + (c/m)\Delta\dot{\theta} + \left(\tfrac{g}{a} - \Omega^2\right)\Delta\theta = 0$ (about $\theta = 0$)

Characteristic equation: $s^2 + \left(\tfrac{c}{m}\right)s + \left(\tfrac{g}{a} - \Omega^2\right) = 0 \quad \Rightarrow \quad s_{1,2} = -\left(\tfrac{c}{2m}\right) \pm \sqrt{\left(\tfrac{c}{2m}\right)^2 - \left(\tfrac{g}{a} - \Omega^2\right)}$

Case 1: $\tfrac{g}{a} > \Omega^2$ and $\left(\tfrac{c}{2m}\right)^2 > \left(\tfrac{g}{a} - \Omega^2\right)$

$$s_{1,2} = -\left(\tfrac{c}{2m}\right) \pm \sqrt{\left(\tfrac{c}{2m}\right)^2 - \left(\tfrac{g}{a} - \Omega^2\right)}$$

Characteristic roots are ***real-valued*** and ***negative***. Solution to the approximate, linear equation of motion is ***over-damped*** and ***stable***

Case 2: $\tfrac{g}{a} > \Omega^2$ and $\left(\tfrac{c}{2m}\right)^2 < \left(\tfrac{g}{a} - \Omega^2\right)$

$$s_{1,2} = -\left(\tfrac{c}{2m}\right) \pm j\sqrt{\left(\tfrac{g}{a} - \Omega^2\right) - \left(\tfrac{c}{2m}\right)^2}$$

Characteristic roots are ***complex-valued*** with ***negative real parts***. Solution to the approximate, linear equation of motion is ***under-damped*** and ***stable***

Case 3: $\Omega^2 > \tfrac{g}{a}$

$$s_{1,2} = -\left(\tfrac{c}{2m}\right) \pm \sqrt{\left(\tfrac{c}{2m}\right)^2 + \left(\Omega^2 - \tfrac{g}{a}\right)}$$

Characteristic roots are ***real-valued*** with ***one positive*** and ***one negative.*** Solution to the approximate, linear equation of motion is ***unstable***