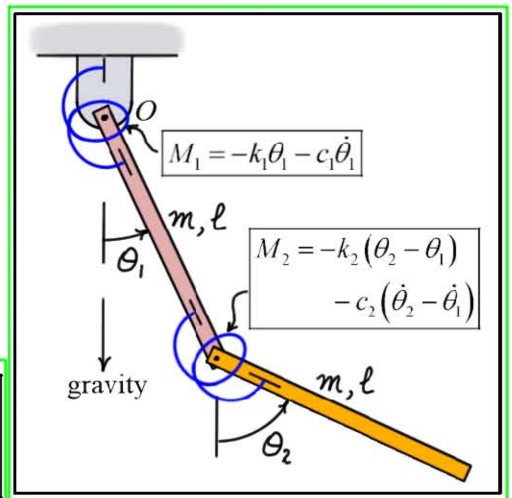
Example #26 – Intermediate Dynamics: (Natural Frequencies, Mode Shapes)

Given:

Linearized equations of motion of the double pendulum with adjoining springs and no damping about $\theta_1 = \theta_2 = 0$.

$$[M]\begin{Bmatrix}\Delta\ddot{\theta}_1\\ \Delta\ddot{\theta}_2\end{Bmatrix} + [K]\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$



where

$$[M] = \begin{bmatrix}\frac{4}{3}m\ell^2 & \frac{1}{2}m\ell^2\\ \frac{1}{2}m\ell^2 & \frac{1}{3}m\ell^2\end{bmatrix} \qquad [K] = \begin{bmatrix}k_1 + k_2 + \frac{3}{2}mg\ell & -k_2\\ -k_2 & k_2 + \frac{1}{2}mg\ell\end{bmatrix}$$

$mg = 5$ (lb) $\quad \ell = 2$ (ft) $\quad k_1 = 1000$ (ft-lb/rad) $\quad k_2 = 300$ (ft-lb/rad)

Find:

Natural frequencies and ***mode shapes*** of the system.

Solution:

$$[M] = \begin{bmatrix}0.828157 & 0.310559\\ 0.310559 & 0.207039\end{bmatrix} \qquad [K] = \begin{bmatrix}1315 & -300\\ -300 & 305\end{bmatrix}$$

$$\det([K] - \lambda[M]) = \begin{bmatrix}(1315 - 0.828157\lambda) & -(300 + 0.310559\lambda)\\ -(300 + 0.310559\lambda) & (305 - 0.207039\lambda)\end{bmatrix}$$
$$= (0.075014)\lambda^2 - (711.18)\lambda + 311075$$

Characteristic equation: $(0.075014)\lambda^2 - (711.18)\lambda + 311075 = 0$ Roots: $\lambda_{1,2} = \begin{Bmatrix}459.697\\ 9020.93\end{Bmatrix}$

Natural Frequencies: $\omega_{1,2} = \sqrt{\lambda_{1,2}} = \begin{Bmatrix}21.4405 \text{ (r/s)}\\ 94.9786 \text{ (r/s)}\end{Bmatrix}$ $\quad f_{1,2} = \frac{\omega_{1,2}}{2\pi} = \begin{Bmatrix}3.4124 \text{ (Hz)}\\ 15.1163 \text{ (Hz)}\end{Bmatrix}$

Mode 1: $f_1 = 3.41$ (Hz)

$$([K] - \lambda_1[M])\{u_1\} = \begin{bmatrix}934.299 & -442.763\\ -442.763 & 209.825\end{bmatrix}\begin{Bmatrix}u_{11}\\ u_{12}\end{Bmatrix} = \{0\}$$

Setting $u_{11} = 1$ and solving either equation for u_{12} gives $u_{12} = 2.110$ $\Rightarrow$ $\{u_1\} = \begin{Bmatrix}1\\ 2.110\end{Bmatrix}$

Mode 2: $f_2 = 15.1$ (Hz)

$$([K] - \lambda_2[M])\{u_2\} = \begin{bmatrix}-6155.75 & -3101.53\\ -3101.53 & -1562.68\end{bmatrix}\begin{Bmatrix}u_{21}\\ u_{22}\end{Bmatrix} = \{0\}$$

Setting $u_{21} = 1$ and solving either equation for u_{22}: $\Rightarrow$ $u_{22} = -1.985$ $\Rightarrow$ $\{u_2\} = \begin{Bmatrix}1\\ -1.985\end{Bmatrix}$