Introductory Control Systems
Performance Indices for Closed-Loop Control

References:

1. D. Graham & R.C. Lathrop, "The Synthesis of Optimal Transient Response," AIEE Trans, Vol 72, No 273, 1953.
2. S. M. Shinners, Modern Control System Theory and Application, Addison-Wesley, 1972
3. R. C. Dorf & R. H. Bishop, Modern Control Systems, 12th edition, Prentice-Hall, Inc, 2010

To ***optimize*** the performance of a closed-loop control system, consider adjusting the parameters of the control system to ***maximize*** or ***minimize some performance index***. Some ***common*** performance indices are defined as follows.

$$ISE = \int_0^T e^2(t)\,dt \qquad IAE = \int_0^T |e(t)|\,dt \qquad ITSE = \int_0^T t e^2(t)\,dt \qquad ITAE = \int_0^T t|e(t)|\,dt$$

Each of the indices are calculated over some interval of time $0 \le t \le T$. The time T is chosen to ***span*** much of the ***transient response*** of the system, so a reasonable choice is T_s the system's settling time. The first two indices ***weight*** the error ***equally*** over the entire time interval, while the last two give ***higher weight*** to the error at later times. Hence, using the latter two indices does ***not penalize*** the system for having ***large initial error***.

<u>ITAE Optimal Response for a Step Input</u>

The references (listed above) indicate that ***ITAE optimal response*** to a ***step input*** can be achieved for systems with transfer functions of the form

$$T(s) = \frac{b_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

The optimal form of the denominator for systems of orders two through six are shown in the table below. These forms are also given in Table 5.5 in Shinners and Table 5.6 in Dorf & Bishop. Note this ***form*** of transfer function has ***zero steady-state error*** to a ***step input***.

Order	Form of Denominator
2	$s^2 + 1.4\omega_n s + \omega_n^2$
3	$s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3$
4	$s^4 + 2.1\omega_n s^3 + 3.4\omega_n^2 s^2 + 2.7\omega_n^3 s + \omega_n^4$
5	$s^5 + 2.8\omega_n s^4 + 5.0\omega_n^2 s^3 + 5.5\omega_n^3 s^2 + 3.4\omega_n^4 s + \omega_n^5$
6	$s^6 + 3.25\omega_n s^5 + 6.60\omega_n^2 s^4 + 8.6\omega_n^3 s^3 + 7.45\omega_n^4 s^2 + 3.95\omega_n^5 s + \omega_n^6$
7	$s^7 + 4.475\omega_n s^6 + 10.42\omega_n^2 s^5 + 15.08\omega_n^3 s^4 + 15.54\omega_n^4 s^3 + 10.64\omega_n^5 s^2 + 4.58\omega_n^6 s + \omega_n^7$
8	$s^8 + 5.20\omega_n s^7 + 12.80\omega_n^2 s^6 + 21.60\omega_n^3 s^5 + 25.75\omega_n^4 s^4 + 22.20\omega_n^5 s^3 + 13.30\omega_n^6 s^2 + 5.15\omega_n^7 s + \omega_n^8$

ITAE Optimal Response for a Ramp Input

The references indicate that ***ITAE optimal response*** to a ***ramp input*** can be achieved for systems with transfer functions of the form

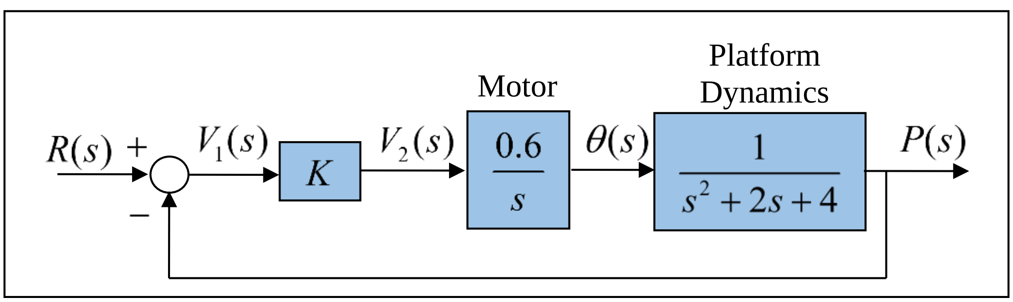
$$T(s) = \frac{b_1 s + b_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

The optimal form of the denominator for these systems of orders two through six are shown in the table below. The forms shown here are also given in Table 5.6 in Shinners and Table 5.7 in Dorf & Bishop. Note this ***form*** of transfer function has ***zero steady-state error*** to a ***ramp input***.

Order	Form of Denominator
2	$s^2 + 3.2\omega_n s + \omega_n^2$
3	$s^3 + 1.75\omega_n s^2 + 3.25\omega_n^2 s + \omega_n^3$
4	$s^4 + 2.41\omega_n s^3 + 4.93\omega_n^2 s^2 + 5.14\omega_n^3 s + \omega_n^4$
5	$s^5 + 2.19\omega_n s^4 + 6.5\omega_n^2 s^3 + 6.3\omega_n^3 s^2 + 5.24\omega_n^4 s + \omega_n^5$
6	$s^6 + 6.12\omega_n s^5 + 13.42\omega_n^2 s^4 + 17.16\omega_n^3 s^3 + 14.14\omega_n^4 s^2 + 6.76\omega_n^5 s + \omega_n^6$

Example

As presented in previous notes, proportional position control of a particular space platform can be expressed by the block diagram below. The open-loop platform dynamics is ***second order*** and ***under-damped***.



Question:

Can a value of gain K be chosen to give the closed-loop system ITAE optimal response to a step input?

Analysis:

The closed-loop transfer function for this system is

$$\frac{P}{R}(s) = \frac{0.6K}{s^3 + 2s^2 + 4s + 0.6K}$$

This transfer function is of the proper form for ***ITAE optimal step response*** so long as the characteristic equation can be put into ***optimal form***. Using the results presented in the table above, the optimal requirement for this third-order system can be written as follows.

$$\boxed{s^3 + 2s^2 + 4s + 0.6K = s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3}$$

Unfortunately, equating the coefficients of the two polynomials gives two ***different values*** for ω_n. So, it is ***not possible*** to adjust K to force the characteristic equation to have the proper third-order form.

Suppose, however, the ***damping coefficient*** in the platform dynamics transfer function can be altered. In this case, the optimal requirement becomes

$$\boxed{s^3 + cs^2 + 4s + 0.6K = s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3}$$

Equating the coefficients of these two polynomials gives $\omega_n = 1.364 \text{ (r/s)}$, $c = 2.39$, and $K = 4.23$. Note that the original damping coefficient was $c = 2$, so this result ***requires*** the ***platform damping*** to be ***increased*** to get optimal step response.

Following this same approach, one could also consider changing the ***platform stiffness***. Either approach is acceptable if the changes to the physical system can be realized. It is always beneficial to consider changes to the ***physical system*** and to the ***control system*** when optimizing system performance.