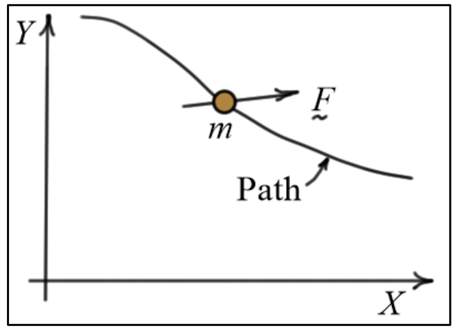
**Elementary Dynamics**
**Principle of Work and Energy**




Recall that Newton's second law for a particle can be written as $\underset{\sim}{F} = m\underset{\sim}{a}$, where $\underset{\sim}{F}$ is the resultant force acting on the particle and $\underset{\sim}{a}$ is its acceleration. Recall also that the force and the acceleration are generally not tangent to its path, except in the case of rectilinear motion.

An integrated form of Newton's second law can be developed relating the ***work done*** on the particle to the change in its ***kinetic energy*** as follows:

***Derivation***:

Taking the ***scalar product*** of both sides of Newton's law with the ***velocity*** $\underset{\sim}{v}$ gives

$$\underset{\sim}{F} \cdot \underset{\sim}{v} = m\left(\underset{\sim}{a} \cdot \underset{\sim}{v}\right) = m\left(\frac{d\underset{\sim}{v}}{dt} \cdot \underset{\sim}{v}\right) = \tfrac{1}{2} m \frac{d}{dt}\left(\underset{\sim}{v} \cdot \underset{\sim}{v}\right)$$

***Integrating*** both sides of this equation with respect to time gives

$$\int_{t_1}^{t_2} \left(\underset{\sim}{F} \cdot \underset{\sim}{v}\right) dt = \tfrac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt}\left(\underset{\sim}{v} \cdot \underset{\sim}{v}\right) dt = \tfrac{1}{2} m \left(\underset{\sim}{v} \cdot \underset{\sim}{v}\right)\Big|_{t_1}^{t_2} = K_2 - K_1 = \Delta K.E.$$

Here,

$$\int_{t_1}^{t_2} \left(\underset{\sim}{F} \cdot \underset{\sim}{v}\right) dt = \int_{t_1}^{t_2} \left(\underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}\right) dt = \oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = U_{1\to 2} \quad \begin{cases} \text{Work done by forces} \\ \text{moving particle from} \\ \text{position 1 to position 2} \end{cases}$$

Note that in the last equation, $\underset{\sim}{F} \cdot \underset{\sim}{v}$ represents the ***power*** of the resultant force $\underset{\sim}{F}$.

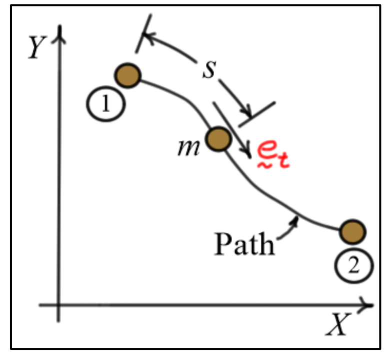
The above results can be generalized to a ***system of particles*** to give

$$\boxed{\sum U_{1\to 2} = \sum \Delta K.E.} \quad \text{or} \quad \boxed{\sum K_1 + \sum U_{1\to 2} = \sum K_2}$$

Here $\sum U_{1\to 2}$ represents the ***total work done*** by all forces acting on the system, and $\sum \Delta K.E.$ represents the ***sum of the changes in kinetic energies*** of all the particles in the system.

## Work Done by Common Forces

### Forces Tangent to Path:

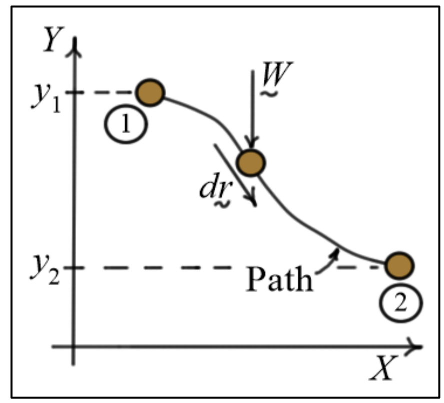



As a particle moves from position 1 to position 2, the work done by a force $\underset{\sim}{F}_t$ ***tangent*** to the path of motion can be written as

$$U_{1\to 2} = \oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{s_1}^{s_2} \left(F_t \underset{\sim}{e}_t\right) \cdot \left(ds\right)\underset{\sim}{e}_t = \int_{s_1}^{s_2} F_t ds$$

If the magnitude $F_t$ is ***constant***, then the integration process gives $U_{1\to 2} = F_t\left(s_2 - s_1\right) = F_t \Delta s$.

### Weight Forces:



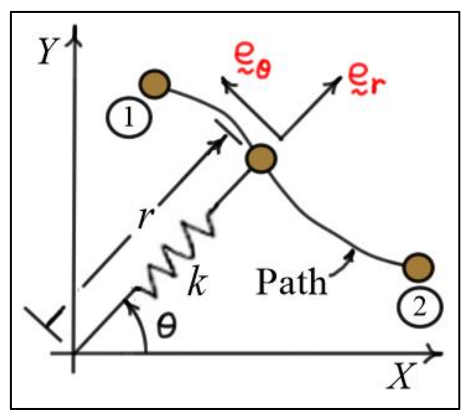

As a particle moves from position 1 to position 2, the work done by its ***weight force*** $\underset{\sim}{W} = -mg\,\underset{\sim}{j}$ can be written as

$$U_{1\to 2} = \oint \underset{\sim}{W} \cdot d\underset{\sim}{r} = \oint \left(-mg\,\underset{\sim}{j}\right) \cdot \left(dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j}\right) = \int_{y_1}^{y_2} -mg\,dy$$

$$U_{1\to 2} = -mg\int_{y_1}^{y_2} dy = -mg\left(y_2 - y_1\right) = -mg\Delta y$$

The work is ***positive*** when the particle moves ***down*** and ***negative*** when it moves ***up***.

### Linear Spring Forces:




As a particle moves from position 1 to position 2, the work done by a spring force $\underset{\sim}{F}_s = -k\left(r - r_o\right)\underset{\sim}{e}_r$ can be written as

$$U_{1\to 2} = \oint \underset{\sim}{F}_s \cdot d\underset{\sim}{r} = \oint -k\left(r - r_o\right)\underset{\sim}{e}_r \cdot \left(dr\,\underset{\sim}{e}_r + r d\theta\,\underset{\sim}{e}_\theta\right)$$

$$U_{1\to 2} = \int_{r_1}^{r_2} -k\left(r - r_o\right)dr = \int_{e_1}^{e_2} -ke\,de \qquad \left\{e = \left(r - r_o\right) = \text{elongation}\right.$$

$$\Rightarrow \boxed{U_{1\to 2} = -\tfrac{1}{2}k\left(e_2^2 - e_1^2\right)}$$