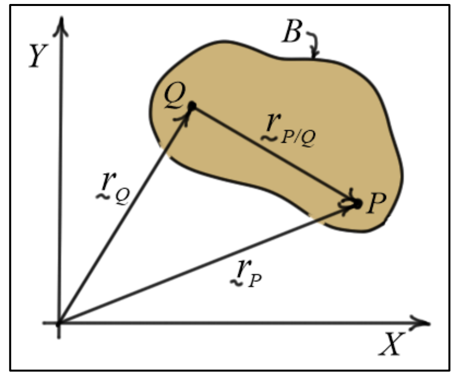
Elementary Dynamics
Relative Velocity of Two Points Fixed on a Rigid Body



The figure depicts a rigid body B moving in two dimensions. The two points P and Q are ***fixed*** on B. At any instant of time, the position vector of P can be written as

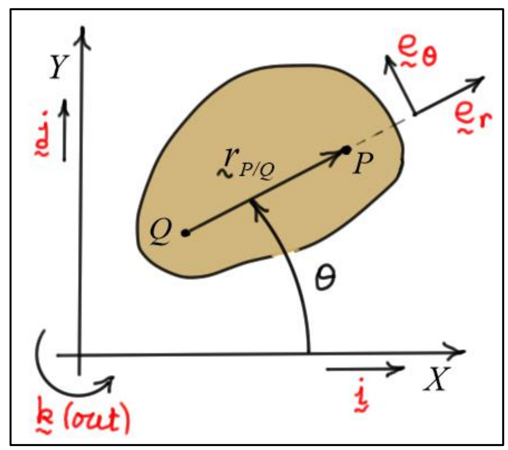
$$\boxed{\underset{\sim}{r}_P = \underset{\sim}{r}_Q + \underset{\sim}{r}_{P/Q}}$$

Here, $\underset{\sim}{r}_{P/Q}$ is called the position vector of P ***relative*** to Q.

The ***velocities*** of P and Q can be related to each other by ***differentiating*** the above equation.

$$\frac{d}{dt}\left(\underset{\sim}{r}_P\right) = \frac{d}{dt}\left(\underset{\sim}{r}_Q\right) + \frac{d}{dt}\left(\underset{\sim}{r}_{P/Q}\right) \quad \text{or} \quad \boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \frac{d}{dt}\left(\underset{\sim}{r}_{P/Q}\right)}$$

Note because the body is rigid, the position vector $\underset{\sim}{r}_{P/Q}$ has ***constant length***. However, the derivative of $\underset{\sim}{r}_{P/Q}$ is ***not zero***, because it will ***change direction*** as the body rotates.



To calculate the derivative of $\underset{\sim}{r}_{P/Q}$, consider the figure at the right. Given $\underset{\sim}{e}_r$ is a unit vector pointed from Q towards P, the position vector $\underset{\sim}{r}_{P/Q}$ can be written as

$$\boxed{\underset{\sim}{r}_{P/Q} = L\underset{\sim}{e}_r}$$

Here L represents the distance from Q to P. The derivative of $\underset{\sim}{r}_{P/Q}$ can be calculated as follows.

$$\frac{d}{dt}\left(\underset{\sim}{r}_{P/Q}\right) = \frac{d}{dt}\left(L\underset{\sim}{e}_r\right) = L\dot{\underset{\sim}{e}}_r = L\left(\dot{\theta}\underset{\sim}{k}\right)\times\underset{\sim}{e}_r = \dot{\theta}\underset{\sim}{k}\times L\underset{\sim}{e}_r = \underset{\sim}{\omega}\times\underset{\sim}{r}_{P/Q} = L\dot{\theta}\underset{\sim}{e}_\theta$$

In this equation, $\underset{\sim}{\omega} \triangleq \dot{\theta}\underset{\sim}{k}$ is the ***angular velocity*** of the body. Combining this result with the boxed equation above gives

$$\boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \left(\underset{\sim}{\omega}\times\underset{\sim}{r}_{P/Q}\right) = \underset{\sim}{v}_Q + \underset{\sim}{v}_{P/Q}} \quad \text{with} \quad \boxed{\underset{\sim}{v}_{P/Q} \triangleq \underset{\sim}{\omega}\times\underset{\sim}{r}_{P/Q}}$$

This last equation defines $\underset{\sim}{v}_{P/Q}$ the ***velocity*** of P ***relative*** to Q. This equation is used to relate the velocity of ***two points fixed*** on the ***same rigid body***.