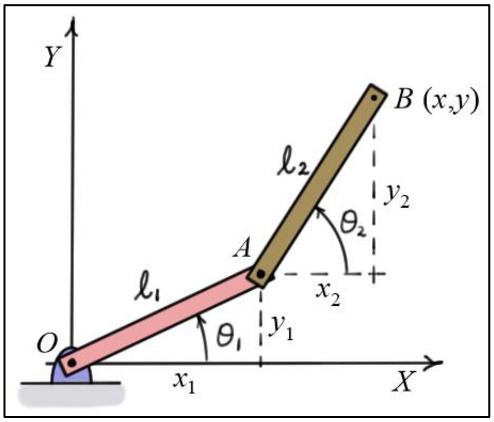
Elementary Engineering Mathematics
Application of Trigonometric Functions in Mechanical Engineering: Part II



Problem: Find the coordinates of the endpoint of a two-link planar robot arm.

Given: The lengths of the links *OA* and *AB* and the angles θ_1 and θ_2.

Find: The *XY* coordinates of the end-point *B*.

Solution:

The coordinates of *B* may be found by adding the coordinates of *A* ***relative to*** *O* and the coordinates of *B* ***relative to*** *A*.

$\boxed{x = x_1 + x_2 = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2)}$ and $\boxed{y = y_1 + y_2 = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2)}$

Example 1:

Given: The lengths and angles of a two link planar robot are $\ell_1 = 3$ (ft), $\ell_2 = 2$ (ft), $\theta_1 = 30$ (deg), and $\theta_2 = 60$ (deg).

Find: The Cartesian coordinates *x* and *y* of *B* using a) a calculator, and b) the values listed above for commonly used angles.

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$\boxed{x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \cos(30)\right) + \left(2 \times \cos(60)\right) = 2.5981 + 1 = 3.5981 \text{ (ft)}}$$

$$\boxed{y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \sin(30)\right) + \left(2 \times \sin(60)\right) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}}$$

b) Using the values for commonly used angles:

$$\boxed{x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \tfrac{\sqrt{3}}{2}\right) + \left(2 \times \tfrac{1}{2}\right) = 2.5981 + 1 = 3.5981 \text{ (ft)}}$$

$$\boxed{y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \tfrac{1}{2}\right) + \left(2 \times \tfrac{\sqrt{3}}{2}\right) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}}$$

Example 2:

Given: The lengths and angles of a two link planar robot are $\ell_1 = 3$ (ft), $\ell_2 = 2$ (ft),

$\theta_1 = 30$ (deg), and $\theta_2 = 120$ (deg).

Find: The Cartesian coordinates x and y of B using a) a calculator, and b) the values listed above for commonly used angles.

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$\boxed{x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \cos(30)\right) + \left(2 \times \cos(120)\right) = 2.5981 - 1 = 1.5981 \text{ (ft)}}$$

$$\boxed{y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \sin(30)\right) + \left(2 \times \sin(120)\right) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}}$$

b) Using the values for commonly used angles: Note first that $\boxed{120 = 180 - 60 \text{ (deg)}}$, so

$$\boxed{\cos(120) = -\cos(60) = -\tfrac{1}{2}} \quad \text{and} \quad \boxed{\sin(120) = \sin(60) = \tfrac{\sqrt{3}}{2}}$$

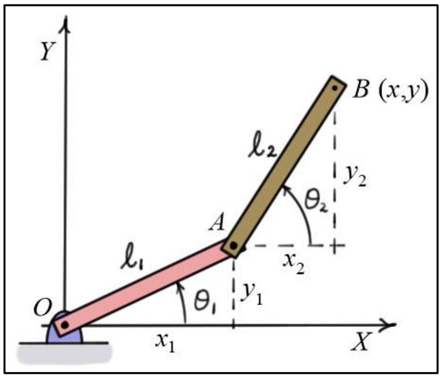
$$\boxed{x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \tfrac{\sqrt{3}}{2}\right) + \left(2 \times (-\tfrac{1}{2})\right) = 2.5981 - 1 = 1.5981 \text{ (ft)}}$$

$$\boxed{y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \tfrac{1}{2}\right) + \left(2 \times \tfrac{\sqrt{3}}{2}\right) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}}$$

Inverse Problem: Find the angles of the links of the robot arm given the endpoint position.

Given: The XY coordinates of the end point B and the lengths of the links OA and AB.

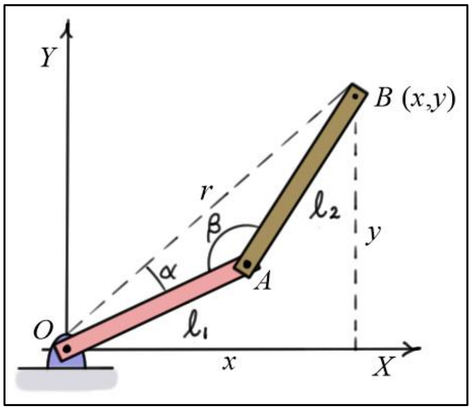
Find: The link angles θ_1 and θ_2.



Solution:

First, calculate the length r using the Pythagorean Theorem.

$$\boxed{r = \sqrt{x^2 + y^2}}$$



Then, apply the ***law of cosines*** to triangle OAB to find angle α.

$$\ell_2^2 = \ell_1^2 + r^2 - 2\ell_1 r \cos(\alpha)$$

or

$$\boxed{\alpha = \cos^{-1}\left(\frac{\ell_1^2 + r^2 - \ell_2^2}{2\ell_1 r}\right)}$$

Finally, apply the ***law of cosines*** again to find angle β.

$$r^2 = \ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos(\beta) \quad \Rightarrow \quad \boxed{\beta = \cos^{-1}\left(\frac{\ell_1^2 + \ell_2^2 - r^2}{2\ell_1\ell_2}\right)}$$

Finally, the link angles can now be found by noting

a) $\tan(\theta_1 + \alpha) = y/x \quad \Rightarrow \quad \boxed{\theta_1 = \tan^{-1}(y/x) - \alpha}$

b) $\theta_2 - \theta_1 = \pi - \beta \quad \Rightarrow \quad \boxed{\theta_2 = \pi - \beta + \theta_1}$

Example 3:

Given: The XY coordinates of the end point B and the lengths of the links OA and AB are

$$x = 1.5 \text{ (ft)},\; y = 3.5 \text{ (ft)},\; \ell_1 = 3 \text{ (ft)}, \text{ and } \ell_2 = 2 \text{ (ft)}.$$

Find: The link angles θ_1 and θ_2.

Solution:

Following the approach outlined above,

a) $\boxed{r = \sqrt{x^2 + y^2} = \sqrt{1.5^2 + 3.5^2} = 3.8079 \text{ (ft)}}$

b) $2^2 = 3^2 + 3.8079^2 - 2 \times 3 \times 3.8079 \times \cos(\alpha)$

$$\Rightarrow \quad \boxed{\alpha = \cos^{-1}\left(\frac{3^2 + 3.8079^2 - 2^2}{2 \times 3 \times 3.8079}\right) = \begin{cases} 31.41 \text{ (deg)} \\ 0.5481 \text{ (rad)} \end{cases}}$$

c) $r^2 = 3^2 + 2^2 - (2 \times 3 \times 2)\cos(\beta) \quad \Rightarrow \quad \boxed{\beta = \cos^{-1}\left(\frac{3^2 + 2^2 - 3.8079^2}{2 \times 3 \times 2}\right) = \begin{cases} 97.18 \text{ (deg)} \\ 1.6961 \text{ (rad)} \end{cases}}$

d) $\tan(\theta_1 + .5481) = 3.5/1.5 \quad \Rightarrow \quad \boxed{\theta_1 = \tan^{-1}(3.5/1.5) - .5481 = \begin{cases} 35.40 \text{ (deg)} \\ 0.6178 \text{ (rad)} \end{cases}}$

$$\theta_2 - \theta_1 = \pi - \beta \quad \Rightarrow \quad \boxed{\theta_2 = \pi - 1.6961 + 0.6178 = \begin{cases} 118.2 \text{ (deg)} \\ 2.0633 \text{ (rad)} \end{cases}}$$

Check:

We can now use the calculated link angles to check the position of the endpoint. Does it match our required position?

$$\boxed{\begin{aligned} x &= \ell_1\cos(\theta_1)+\ell_2\cos(\theta_2) \\ &= \left(3\times\cos(0.6178)\right)+\left(2\times\cos(2.0633)\right)=2.4455-0.9457=1.4998\approx 1.5 \text{ (ft)} \end{aligned}} \checkmark$$

$$\boxed{\begin{aligned} y &= \ell_1\sin(\theta_1)+\ell_2\sin(\theta_2) \\ &= \left(3\times\sin(0.6178)\right)+\left(2\times\sin(2.0633)\right)=1.7377+1.7623=3.5 \text{ (ft)} \end{aligned}} \checkmark$$

Note on calculator usage:

When calculating $\sin^{-1}(\theta)$, $\cos^{-1}(\theta)$ and $\tan^{-1}(\theta)$, your calculator will place the results in specific quadrants as outlined in the table to the right. So, your calculator does not always place the angle into the correct quadrant.

Function	Range	Quadrants
$\sin^{-1}(\theta)$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	I, IV
$\cos^{-1}(\theta)$	$0\le\theta\le\pi$	I, II
$\tan^{-1}(\theta)$	$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	I, IV

Note that in the above example, we used the ***law of cosines*** (and hence $\cos^{-1}(\theta)$) to calculate the angles of the triangle *OAB*, and our calculator gave angles in the range $0\le\theta\le\pi$. What if we had used the ***law of sines*** to calculate the angle β?

Law of Sines:

$$\frac{\sin(\beta)}{r}=\frac{\sin(\alpha)}{\ell_2}$$

$$\Rightarrow \boxed{\beta=\sin^{-1}\left(r\sin(\alpha)/\ell_2\right)=\sin^{-1}\left(3.8079\times\sin(0.5481)/2\right)=\begin{cases}82.79 \text{ (deg)}\\ 1.4449 \text{ (rad)}\end{cases}}$$

Note this is ***not*** the correct result. As we know from our work above, the correct result is in the ***second quadrant***. So, $\beta=\pi-1.4449=1.6967 \text{ (rad)}$. This is very close to the result found above.

Elbow-down and Elbow-up Positions

Note that the above answers could be interpreted in two ways – the elbow-down or elbow-up positions as illustrated in the following diagrams. The numerical results are the same but with a different physical interpretation. Mathematical results must always be interpreted with the physical system in mind.