

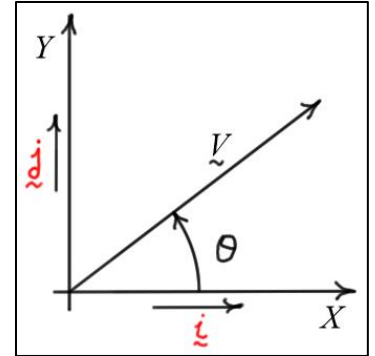
Elementary Engineering Mathematics

Application of Two-Dimensional Vectors – Statics, Mechanics of Materials, Dynamics

Scalars and Vectors

A *scalar* is a quantity represented by a *positive or negative number*. It contains a magnitude (its absolute value) and a sign. They are sometimes called *one-dimensional vectors*, because the sign refers to the direction along a single axis. Examples include length, area, volume, mass, pressure, and temperature.

A two-dimensional (2D) vector is represented by a *magnitude* and a *direction* related to *two reference axes*. Usually, the reference axes (X and Y) are perpendicular to each other. In application, vectors can be categorized as *fixed* or *free*. A *fixed* vector is defined to be anchored at a specific point, whereas *free* vectors can be located anywhere without changing their meaning. Whether a vector is thought to be fixed or free depends on the quantity the vector represents.



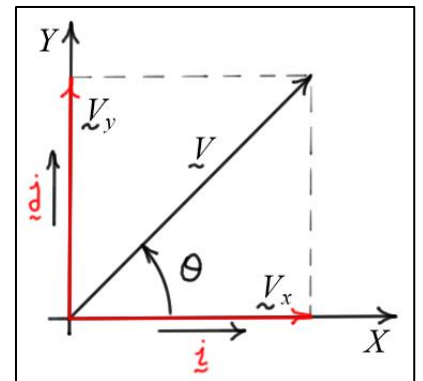
For example, if vector \underline{V} in the diagram represents a *force* acting on some object, it is considered to be a fixed vector, because its point of application is important. In contrast, consider vectors \underline{i} and \underline{j} shown in the diagram. These vectors have unit magnitude and point along the X and Y axes, respectively. They are called *unit vectors* and are used to define directions of interest. Since their point of origin is not important, they are free vectors. The *mathematical representation* of a vector does not indicate whether it is fixed or free, so we must be mindful of this as we use them.

Given: The magnitude $|\underline{V}|$ and direction θ of vector \underline{V}

Find: The X and Y *components* of \underline{V}

Solution:

The diagram shows the X and Y components of \underline{V} labeled as \underline{V}_x and \underline{V}_y . These components are also two-dimensional vectors, and their magnitudes are given by right-triangle trigonometry. Their directions are along \underline{i} and \underline{j} directions, respectively.



Vector \underline{V} is the sum of these two components.

$$\underline{V}_x = |\underline{V}| \cos(\theta) \underline{i}$$

$$\underline{V}_y = |\underline{V}| \sin(\theta) \underline{j}$$

$$\underline{V} = |\underline{V}| \cos(\theta) \underline{i} + |\underline{V}| \sin(\theta) \underline{j}$$

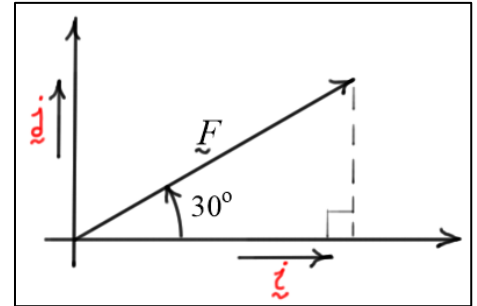
Example 1:

Given: A force \underline{F} has a magnitude $|\underline{F}| = 100$ (lbs) and an angle $\theta = 30$ (deg).

Find: Express the force \underline{F} in terms of the unit vectors \underline{i} and \underline{j} .

Solution:

$$\underline{F} = 100 \cos(30) \underline{i} + 100 \sin(30) \underline{j} = 86.6 \underline{i} + 50 \underline{j} \text{ (lbs)}$$



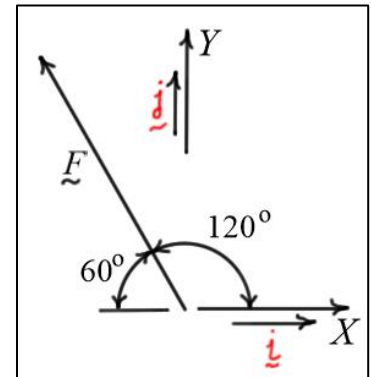
Example 2:

Given: A force \underline{F} has a magnitude $|\underline{F}| = 100$ (lbs) and an angle $\theta = 120$ (deg).

Find: Express the force \underline{F} in terms of the unit vectors \underline{i} and \underline{j} .

Solution:

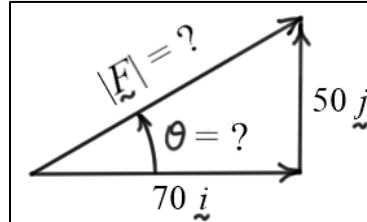
$$\underline{F} = 100 \cos(120) \underline{i} + 100 \sin(120) \underline{j} = -50 \underline{i} + 86.6 \underline{j} \text{ (lbs)}$$



Example 3:

Given: A force $\underline{F} = 70 \underline{i} + 50 \underline{j}$ (lbs).

Find: The magnitude and direction of \underline{F} .



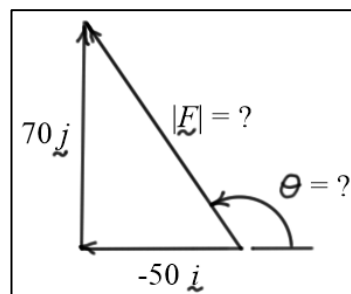
Solution:

$$|\underline{F}| = \sqrt{70^2 + 50^2} \approx 86.0 \text{ (lbs)} \quad \text{and} \quad \theta = \tan^{-1}(50/70) \approx \begin{cases} 35.54 \text{ (deg)} \\ 0.6202 \text{ (rad)} \end{cases}$$

Example 4:

Given: A force $\underline{F} = -50 \underline{i} + 70 \underline{j}$ (lbs).

Find: The magnitude and direction of \underline{F} .



Solution:

$$|\underline{F}| = \sqrt{(-50)^2 + 70^2} \approx 86.0 \text{ (lbs)} \quad \theta = \tan^{-1}(70/-50) \approx \begin{cases} -54.46 + 180 = 125.5 \text{ (deg)} \\ -0.9505 + \pi = 2.191 \text{ (rad)} \end{cases}$$

Vector Addition

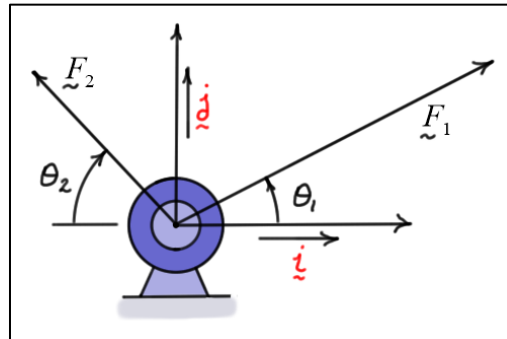
To **add** two or more vectors, simply express them in terms of the same unit vectors, and then add **like components**.

Example 5:

Given: $|\vec{F}_1| = 150$ (lbs), $\theta_1 = 20$ (deg)

$|\vec{F}_2| = 100$ (lbs), $\theta_2 = 60$ (deg)

Find:



a) The **total force** \vec{F} acting on the support in terms of the unit vectors shown.

b) The **magnitude** and **direction** of \vec{F} .

Solution:

a) The total force is the vector sum of the two forces.

$$\vec{F}_1 = 150\cos(20)\vec{i} + 150\sin(20)\vec{j} = 140.95\vec{i} + 51.3\vec{j} \text{ (lbs)}$$

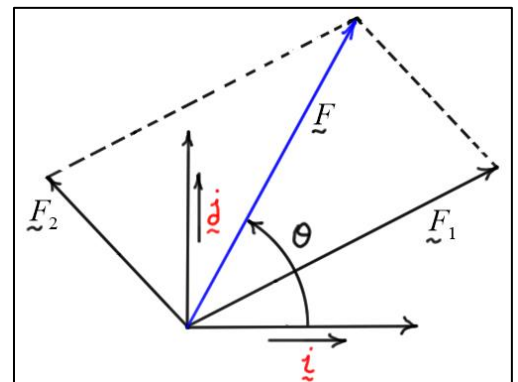
$$\vec{F}_2 = -100\cos(60)\vec{i} + 100\sin(60)\vec{j} = -50\vec{i} + 86.6\vec{j} \text{ (lbs)}$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 \approx (140.95 - 50)\vec{i} + (51.3 + 86.6)\vec{j} \approx 90.95\vec{i} + 137.9\vec{j} \text{ (lbs)}$$

b) $|\vec{F}| \approx \sqrt{90.95^2 + 137.9^2} \approx 165.2$ (lbs)

$$\theta \approx \tan^{-1}(137.9 / 90.95) \approx \begin{cases} 56.59 \text{ (deg)} \\ 0.9877 \text{ (rad)} \end{cases}$$

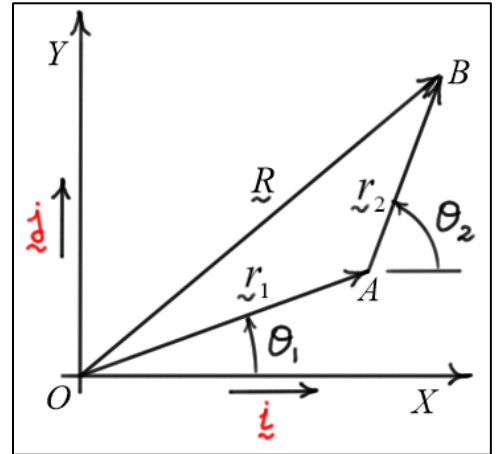
As you can see from the figure on the right, \vec{F}_1 and \vec{F}_2 form the sides of a parallelogram, and the sum \vec{F} forms the diagonal. The observation that vectors can be added geometrically in this way is called the **parallelogram law of addition**. In general, the triangle formed by \vec{F}_1 , \vec{F}_2 , and \vec{F} is a non-right triangle. The lengths and angles within this triangle can be studied using the **laws of cosines** and **sines** as discussed in previous notes.



Example 6:

Given: The lengths and angles of a two link planar robot are $|r_1| = 3$ (ft), $|r_2| = 2$ (ft), $\theta_1 = 20$ (deg), and $\theta_2 = 70$ (deg).

Find: a) the position vector \underline{R} that defines the position of the endpoint of the robot relative to O .
b) the magnitude and direction of \underline{R} .



Solution:

a) The position vector \underline{R} can be calculated by adding the vectors r_1 and r_2 .

$$\begin{aligned}\underline{R} &= r_1 + r_2 = (3\cos(20)\underline{i} + 3\sin(20)\underline{j}) + (2\cos(70)\underline{i} + 2\sin(70)\underline{j}) \\ &\approx (2.8191\underline{i} + 1.0261\underline{j}) + (0.684\underline{i} + 1.8794\underline{j}) \\ &\Rightarrow \underline{R} \approx 3.503\underline{i} + 2.906\underline{j} \text{ (ft)}\end{aligned}$$

b) $|\underline{R}| \approx \sqrt{3.503^2 + 2.906^2} \approx 4.55$ (ft) and $\theta \approx \tan^{-1}(2.906 / 3.503) \approx \begin{cases} 39.7 \text{ (deg)} \\ 0.6925 \text{ (rad)} \end{cases}$

Scalar (Dot) Product

Geometric Definition

The *scalar* (or dot) product of two vectors is defines as follows.

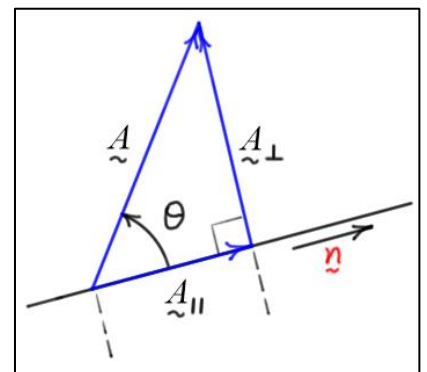
$$\underline{A} \cdot \underline{B} = |\underline{A}| |\underline{B}| \cos(\underline{A}, \underline{B})$$

Here, $\cos(\underline{A}, \underline{B})$ represents the *cosine* of the angle between the tails of the two vectors. If one of the vectors is a *unit vector*, then the scalar product is the *projection* of the vector in the direction of the unit vector.

$$\underline{A} \cdot \underline{n} = |\underline{A}| |\underline{n}| \cos(\theta) = |\underline{A}| \cos(\theta)$$

The *components* of \underline{A} that are *parallel* and *perpendicular* to \underline{n} are

$$\underline{A}_{\parallel} = (\underline{A} \cdot \underline{n})\underline{n} \quad \text{and} \quad \underline{A}_{\perp} = \underline{A} - \underline{A}_{\parallel}$$



Calculation of the Dot Product of Two Vectors

Given two vectors \underline{A} and \underline{B} expressed in terms of a pair of mutually perpendicular unit vectors \underline{i} and \underline{j} , we calculate the dot product as

$$\underline{A} \cdot \underline{B} = (a_x \underline{i} + a_y \underline{j}) \cdot (b_x \underline{i} + b_y \underline{j}) = a_x b_x + a_y b_y$$

The **dot product** of two vectors is **zero** if they are **perpendicular** to each other.

Example 7:

Given: Two vectors, $\underline{A} = 10\underline{i} + 2\underline{j}$ and $\underline{B} = 3\underline{i} + 7\underline{j}$

Find: The angle between the two vectors, θ .

Solution:

We can calculate the angle using the inverse cosine function.

$$\theta = \cos^{-1} \left(\frac{\underline{A} \cdot \underline{B}}{|\underline{A}| |\underline{B}|} \right) = \cos^{-1} \left(\frac{(10 \times 3) + (2 \times 7)}{\sqrt{10^2 + 2^2} \sqrt{3^2 + 7^2}} \right) \approx \cos^{-1} \left(\frac{44}{77.666} \right) \approx \begin{cases} 55.49 \text{ (deg)} \\ 0.9685 \text{ (rad)} \end{cases}$$

Example 8:

Given: A vector, $\underline{A} = 2\underline{i} + 8\underline{j}$, and a unit vector $\underline{n} = \frac{3}{5}\underline{i} + \frac{4}{5}\underline{j}$

Find: a) θ the **angle** between the two vectors

b) the component of vector \underline{A} **parallel** to unit vector \underline{n}

c) the component of vector \underline{A} **perpendicular** to unit vector \underline{n}

Solution:

a) We can calculate the angle using the inverse cosine function as before.

$$\theta = \cos^{-1} \left(\frac{\underline{A} \cdot \underline{n}}{|\underline{A}|} \right) = \cos^{-1} \left(\frac{(2 \times \frac{3}{5}) + (8 \times \frac{4}{5})}{\sqrt{2^2 + 8^2}} \right) \approx \cos^{-1} \left(\frac{7.6}{8.2462} \right) \approx \begin{cases} 22.83 \text{ (deg)} \\ 0.3985 \text{ (rad)} \end{cases}$$

b) The component of \underline{A} parallel to \underline{n} : $\underline{A}_{\parallel} = (\underline{A} \cdot \underline{n})\underline{n} = 7.6 \left(\frac{3}{5}\underline{i} + \frac{4}{5}\underline{j} \right) = 4.56\underline{i} + 6.08\underline{j}$

c) The component of \underline{A} perpendicular to \underline{n} :

$$\underline{A}_{\perp} = \underline{A} - \underline{A}_{\parallel} \approx (2\underline{i} + 8\underline{j}) - (4.56\underline{i} + 6.08\underline{j}) \approx -2.56\underline{i} + 1.92\underline{j}$$

Check: $\underline{A}_{\perp} \cdot \underline{A}_{\parallel} \approx (-2.56\underline{i} + 1.92\underline{j}) \cdot (4.56\underline{i} + 6.08\underline{j}) \approx (-2.56 \times 4.56) + (1.92 \times 6.08) \approx 0$ ✓

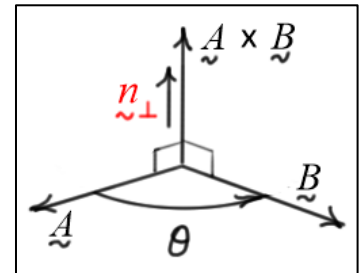
Vector (Cross) Product

Geometric Definition

The **cross** product of two vectors is defined as follows.

$$\underline{A} \times \underline{B} = (|\underline{A}| |\underline{B}| \sin(\underline{A}, \underline{B})) \underline{n}_{\perp}$$

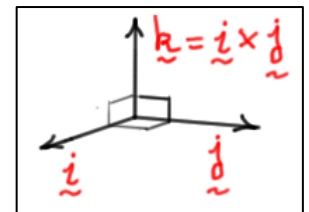
Here, $\sin(\underline{A}, \underline{B})$ is the **cosine** of the angle between the tails of the two vectors, and \underline{n}_{\perp} is a unit vector perpendicular to the plane formed by vectors \underline{A} and \underline{B} . The sense of \underline{n}_{\perp} is defined by the **right-hand-rule**, that is, the right thumb points in the direction of \underline{n}_{\perp} when the fingers of the right hand point from \underline{A} to \underline{B} .



Calculation

Given two vectors \underline{A} and \underline{B} expressed in terms of mutually perpendicular unit vectors \underline{i} and \underline{j} , we calculate the cross product as

$$\underline{A} \times \underline{B} = (a_x \underline{i} + a_y \underline{j}) \times (b_x \underline{i} + b_y \underline{j}) = (a_x b_y - a_y b_x) \underline{k}$$



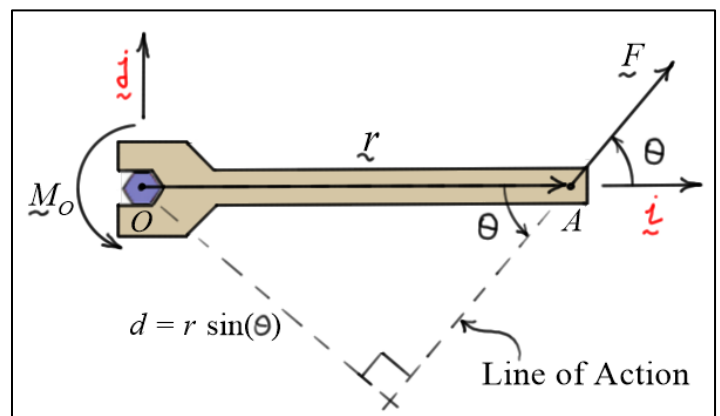
The **cross product** of two vectors is **zero** if they are **parallel** to each other.

This calculation may be calculated from the **determinant form**. Since the cross product of two dimensional vectors has no \underline{i} or \underline{j} components, we write

$$\underline{A} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix} = (a_x b_y - a_y b_x) \underline{k}$$

Moment of a Force – Torque

The **moment** (or **torque**) of a force about a point O is defined as the magnitude of the force ($|\underline{F}|$) multiplied by the **perpendicular distance** from the point to the **line of action** of the force ($d = r \sin(\theta)$). The right-hand-rule defines the direction. So, it can be calculated using the **cross product**.



$$\boxed{\underline{M}_O = \underline{r} \times \underline{F}}$$

Here, \underline{r} is a position vector from O to the line of action of \underline{F} .

Example 9:

Given: A force $\underline{F} = 50\hat{i} + 100\hat{j}$ (lbs) is applied at a point A whose coordinates are $(3,2)$ (ft).

Find: a) \underline{M}_O the moment of the force about O the origin at $(0,0)$

b) d the perpendicular distance from O to the line of action of the force

Solution:

$$\text{a) } \boxed{\underline{M}_O = \underline{r} \times \underline{F} = (3\hat{i} + 2\hat{j}) \times (50\hat{i} + 100\hat{j}) = ((3 \times 100) - (2 \times 50))\hat{k} = 200\hat{k} \text{ (ft-lbs)}}$$

$$\text{b) } \boxed{d = \frac{|\underline{M}_O|}{|\underline{F}|} = \frac{200}{\sqrt{50^2 + 100^2}} \approx \frac{200}{111.80} \approx 1.79 \text{ (ft)}}$$

Example 10:

Given: A force $\underline{F} = 50\hat{i} + 100\hat{j}$ (lbs) is applied at a point A whose coordinates are $(3,-2)$ (ft).

Find: a) \underline{M}_B the moment of the force about point B whose coordinates are $(5,10)$ (ft)

b) d the perpendicular distance from B to the line of action of the force

Solution:

To calculate the moment using the cross product, we must first calculate the position vector that defines the position of A *relative to* B .

$$\text{a) } \boxed{\underline{r} = \underline{r}_{A/B} = \underline{r}_A - \underline{r}_B = (3\hat{i} - 2\hat{j}) - (5\hat{i} + 10\hat{j}) = -2\hat{i} - 12\hat{j} \text{ (ft)}}$$

$$\boxed{\underline{M}_B = \underline{r} \times \underline{F} = (-2\hat{i} - 12\hat{j}) \times (50\hat{i} + 100\hat{j}) = ((-2 \times 100) + (12 \times 50))\hat{k} = 400\hat{k} \text{ (ft-lb)}}$$

$$\text{b) } \boxed{d = \frac{|\underline{M}_B|}{|\underline{F}|} = \frac{400}{\sqrt{50^2 + 100^2}} \approx \frac{400}{111.80} \approx 3.58 \text{ (ft)}}$$