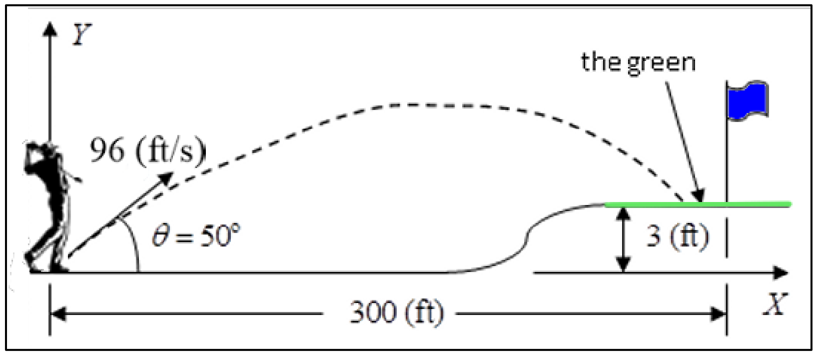
Elementary Engineering Mathematics
Introduction to Complex Numbers

Introduction

Recall that when we calculate the ***roots*** of a ***quadratic equation***, we may get real roots, or we may get a complex conjugate pair. As an example, consider the golf ball trajectory problem we discussed in earlier notes.



To find the times when the ball is ***50 feet*** above the ground $\left(y = 50 \text{ (ft)}\right)$, we solved the quadratic equation $\boxed{16.1t^2 - 73.54t + 50 = 0}$ using the ***quadratic formula*** and found

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)50}}{2(16.1)} \approx 2.2839 \pm 1.4527 \quad \Rightarrow \boxed{t_{1,2} \approx \begin{cases} 0.8312 \approx 0.831 \text{ (s)} \\ 3.7366 \approx 3.74 \text{ (s)} \end{cases}}$$

The ball passes the 50-foot mark on its ***way up*** and on its ***way down***.

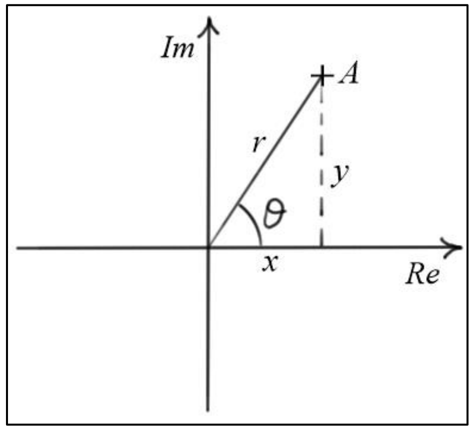
To find the times when the ball is ***100 feet*** above the ground, we solved the equation $\boxed{16.1t^2 - 73.54t + 100 = 0}$ and found

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)100}}{2(16.1)} = \frac{73.54 \pm \sqrt{-1031.87}}{32.2} = \frac{73.54 \pm j\sqrt{1031.87}}{32.2}$$
$$\approx \frac{73.54 \pm j\,32.1227}{32.2} \approx \boxed{2.2839 \pm j0.9976}$$

The result is a ***complex conjugate pair*** $\left(j = \sqrt{-1}\right)$. This occurs because the ball ***never reaches 100 feet***, so ***no real solutions exist***.

Complex Numbers and the Complex Plane

Generally, complex numbers have both ***real*** (*Re*) and ***imaginary*** (*Im*) parts. The diagram shows a complex number A plotted in the ***complex plane***. We can express A using either ***rectangular*** or ***polar*** coordinates.



Rectangular form: $\boxed{A = x + j\,y}$

Polar Form: $\boxed{A = r\,e^{j\theta}}$ or $\boxed{A = r\angle\theta}$

We can relate the rectangular and polar forms using ***right-triangle trigonometry***.

Given the ***rectangular form*** $\boxed{A = x + j\,y}$, we can find the ***polar form*** $\boxed{A = r\,e^{j\theta}}$.

$\boxed{r = \sqrt{x^2 + y^2} = |A|}$ …the magnitude of A

$\boxed{\theta = \tan^{-1}\left(\dfrac{y}{x}\right)}$ …the phase angle of A

Given the ***polar form*** $\boxed{A = r\,e^{j\theta}}$, we can find the ***rectangular form*** $\boxed{A = x + j\,y}$.

$\boxed{x = r\cos(\theta)}$ …the real part of A

$\boxed{y = r\sin(\theta)}$ …the imaginary part of A

Using these results, we can identify Euler's formula:

$$A = x + j\,y = (r\cos(\theta)) + j(r\sin(\theta)) = r(\cos(\theta) + j\sin(\theta)) = r\,e^{j\theta}$$

or

$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$ … ***Euler's formula***

Complex Conjugates

Given a complex number $A = a_1 + j\,a_2$, the complex conjugate of A is defined as

$\boxed{A^* \triangleq a_1 - j\,a_2}$ …the complex conjugate

Operations with Complex Numbers

Addition and Subtraction

Addition and subtraction of complex numbers is most easily done in ***rectangular form***. Given two complex numbers $A = a_1 + j\,a_2$ and $B = b_1 + j\,b_2$, then

$$\boxed{A + B = (a_1 + b_1) + j\,(a_2 + b_2)} \quad \text{and} \quad \boxed{A - B = (a_1 - b_1) + j\,(a_2 - b_2)}$$

If *A* and *B* are given in ***polar form***, it is best to ***convert*** them to rectangular form before adding or subtracting.

Multiplication and Division (Polar Form)

Multiplication and division of complex numbers is most easily done in ***polar form***.

$$\boxed{A \times B = \left(ae^{j\alpha}\right)\left(be^{j\beta}\right) = a\,b\,e^{j(\alpha+\beta)}} \quad \text{and} \quad \boxed{A\,/\,B = \left(ae^{j\alpha}\right)\Big/\left(be^{j\beta}\right) = \left(a/b\right)e^{j(\alpha-\beta)}}$$

$$\boxed{A \times B = \left(a\angle\alpha\right)\left(b\angle\beta\right) = ab\angle(\alpha+\beta)} \quad \text{and} \quad \boxed{A/B = \left(a\angle\alpha\right)\Big/\left(b\angle\beta\right) = \left(a/b\right)\angle(\alpha-\beta)}$$

If *A* and *B* are given in ***rectangular form***, it is usually best to ***convert*** them to polar form before multiplying or dividing.

Multiplication/Division (Rectangular Form)

Multiplication and division of complex numbers can also be done (with a little more work) using ***rectangular form***.

Multiplication

Given two complex numbers $A = a_1 + j\,a_2$ and $B = b_1 + j\,b_2$, then their product is

$$\boxed{A \times B = (a_1 b_1 - a_2 b_2) + j\,(a_1 b_2 + a_2 b_1)} \quad \left(j \times j = -1\right)$$

Note that if *B* is the complex conjugate of *A* $\left(B = A^*\right)$, the product is a real number equal to the square of the magnitude of *A*.

$$\boxed{A \times A^* = (a_1^2 + a_2^2) + j\,(a_1 a_2 - a_2 a_1) = a_1^2 + a_2^2 = |A|^2}$$

Division

To compute the ratio of A and B is a little more involved. To ensure that the imaginary parts appear only in the numerator, we must make use of the complex conjugate.

$$\frac{A}{B}=\frac{a_1+j\,a_2}{b_1+j\,b_2}=\left(\frac{a_1+j\,a_2}{b_1+j\,b_2}\right)\cdot\left(\frac{b_1-j\,b_2}{b_1-j\,b_2}\right)=\frac{(a_1b_1+a_2b_2)+j(a_2b_1-a_1b_2)}{b_1^2+b_2^2}$$

Example #1

Given: $A=5+j\,10$ Find: the polar form of A

Solution: $\boxed{r=|A|=\sqrt{5^2+10^2}=\sqrt{125}\approx 11.2}$

$\boxed{\theta=\tan^{-1}(10/5)\approx 1.107\text{ (rad)}\approx 63.4\text{ (deg)}}$

Example #2

Given: $A=-5+j\,10$ Find: the polar form of A

Solution: $\boxed{r=|A|=\sqrt{5^2+10^2}=\sqrt{125}\approx 11.2}$

$\boxed{\theta=\tan^{-1}(10/-5)\approx -1.107+\pi\approx 2.03\text{ (rad)}\approx 117\text{ (deg)}}$

Example #3

Given: $A=5+j\,10$ Find: $A\times A^*$

Solution: $\boxed{A\times A^*=(5+j10)\times(5-j10)=5^2+10^2=125}$

Example #4

Given: $A=5+j\,10$ and $B=3-j\,8$ Find: $A+B$, $A\times B$ and A/B

Solution: $\boxed{A+B=(5+j\,10)+(3-j\,8)=8-j\,2}$

$$A\times B=(5+j\,10)\times(3-j\,8)=\big((5\times 3)-(-8\times 10)+j\big((3\times 10)-(5\times 8)\big)$$

$$\Rightarrow\ \boxed{A\times B=95-j10}$$

$$A/B=\frac{(5+j\,10)}{(3-j\,8)}=\frac{(5+j\,10)\times(3+j8)}{(3-j\,8)\times(3+j8)}=\frac{\big((15-80)+j(30+40)\big)}{3^2+8^2}=\frac{-65+j\,70}{73}$$

$$\Rightarrow\ \boxed{A/B=-0.89+j\,0.959}$$

Example #5

Given: $A = 5e^{j(\pi/3)}$ and $B = 8e^{j(-\pi/6)}$ Find: $A \times B$ and A/B

Solution:

$$A \times B = \left(5e^{j(\pi/3)}\right) \times \left(8e^{j(-\pi/6)}\right) = (5 \times 8)\, e^{j(\pi/3 + (-\pi/6))} = 40e^{j(\pi/6)}$$

$$A/B = \left(5e^{j(\pi/3)}\right) / \left(8e^{j(-\pi/6)}\right) = (5/8)\, e^{j(\pi/3 - (-\pi/6))} = 0.625e^{j(\pi/2)}$$

Example #6

Given: $A = 5e^{j(\pi/3)}$ and $B = 8e^{j(-\pi/6)}$ Find: $A + B$

Solution: We first ***convert*** the polar forms to rectangular forms, then add

$$A = 5e^{j(\pi/3)} = 5\left(\cos(\pi/3) + j\sin(\pi/3)\right) \approx 2.5 + j4.33$$

$$B = 8e^{j(-\pi/6)} = 8\left(\cos(-\pi/6) + j\sin(-\pi/6)\right) \approx 6.928 - j4$$

$$A + B \approx 9.43 + j0.33$$

If necessary, this result can then be converted back to polar form as described above.