

Elementary Engineering Mathematics

Application of Complex Numbers in Electric Circuits

Impedance in AC (Alternating Current) Circuits

In AC circuits, the steady-state voltages and currents are nearly *sinusoidal*. They alternate at some *frequency* ω (rad/sec) and have both *magnitude* and *phase*. We can analyze these signals using *complex numbers* and a *complex form* of *Ohm's law*. This form of Ohm's law relates the *sinusoidal signals* to the *impedances* of the circuit elements. The *impedances* are expressed as *complex numbers* and are measured in *ohms* (Ω). Impedance in an AC circuit is analogous to resistance in a DC (Direct Current) circuit.

Resistor

The impedance of a resistor is its resistance R , $Z_R = R + j0 = R e^{j(0)} = R \angle 0^\circ$. The impedance of a resistor is the same at all frequencies (ω).

Capacitor

The impedance of a capacitor is $Z_C = -j/\omega C = (1/\omega C) e^{j(-90^\circ)} = (1/\omega C) \angle (-90^\circ)$. Here, ω is the *frequency* of the signals in (rad/s) and C is the *capacitance* measured in *farads* (f). Often C is provided in *micro-farads* (μf). A micro-farad is 10^{-6} farads.

Inductor

The impedance of an inductor is $Z_L = j\omega L = (\omega L) e^{j(90^\circ)} = (\omega L) \angle (90^\circ)$. Here, L is the *inductance* measured in *henrys* (h). Often L is provided in *millihenrys* (mh). A millihenry is 10^{-3} henrys. As before, ω is the *frequency* of the signals in (rad/s).

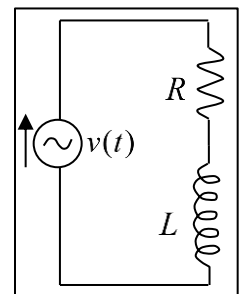
Example #1

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RL series circuit with $R = 100$ (ohms) and $L = 100$ (mh). The impedance of the circuit is

$$Z = Z_R + Z_L \quad \text{and the frequency is } \omega = 120\pi \text{ (rad/s)}.$$

Find: Find the complex impedance Z .



Solution:

$$Z = Z_R + Z_L = 100 + j(120\pi)(0.1) \Rightarrow Z \approx 100 + j37.7 \text{ (ohms)}$$

or

$$Z \approx 106.9 e^{j(20.7^\circ)} \approx 106.9 \angle (20.7^\circ) \text{ (ohms)}$$

Example #2

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RC series circuit with $R = 100$ (ohms) and $C = 20$ (μf). The impedance of the circuit is

$$Z = Z_R + Z_C \text{ and the frequency is } \omega = 120\pi \text{ (rad/s)}.$$

Find: Find the complex impedance Z .

Solution:

$$Z = Z_R + Z_C = 100 - j/(120\pi)(20 \times 10^{-6}) \Rightarrow Z \approx 100 - j132.6 \text{ (ohms)}$$

or

$$Z \approx 166.1 e^{j(-52.98^\circ)} \approx 166.1 \angle (-52.98^\circ) \text{ (ohms)}$$

Example #3

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RLC series circuit with $R = 100$ (ohms), $L = 100$ (mh), and $C = 20$ (μf). The impedance of the

circuit is $Z = Z_R + Z_L + Z_C$ and the frequency is $\omega = 120\pi$ (rad/s).

Find: Find the complex impedance Z .

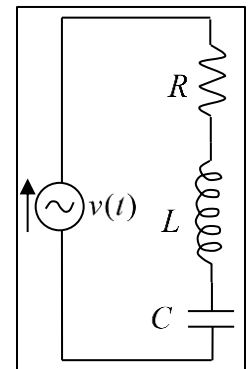
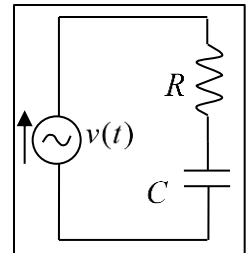
Solution: (using the results from the examples above)

$$Z = Z_R + Z_L + Z_C \approx 100 + j(37.7 - 132.6) \Rightarrow Z \approx 100 - j94.9 \text{ (ohms)}$$

or

$$Z \approx 137.9 e^{j(-43.5^\circ)} \approx 137.9 \angle (-43.5^\circ) \text{ (ohms)}$$

Note: The *impedance* of *capacitors* and *inductors* is a *function of frequency*, whereas the impedance of a resistor is the same at all frequencies.



Complex Form of Ohm's Law

To find the currents in the above circuits, we use a complex form of Ohm's law which states that the **voltage drop across an impedance** is equal to the product of the complex current and complex impedance. That is, $V = IZ$. Rewriting this equation, we can find the current

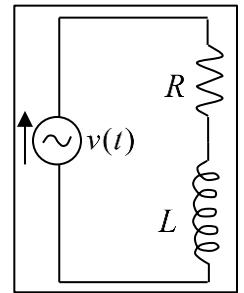
$$I = V/Z.$$

Example #4

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RL series circuit with $R = 100$ (ohms) and $L = 100$ (mh). The impedance of the circuit is

$$Z = Z_R + Z_L \text{ and the frequency is } \omega = 120\pi \text{ (rad/s)}.$$



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, the impedance is $Z = Z_R + Z_L \approx 100 + j37.7$ (ohms)

or

$$Z \approx 106.9e^{j(20.7^\circ)} \approx 106.9 \angle (20.7^\circ) \text{ (ohms)}$$

The given voltage $v(t)$ can be written in complex form as $V = 110e^{j(0^\circ)} = 110 \angle (0^\circ)$.

So, the complex current is

$$I = V/Z \approx 110e^{j(0^\circ)} / 106.9e^{j(20.7^\circ)} \approx 1.029e^{j(-20.7^\circ)} \text{ (amps)}$$

and

$$i(t) \approx 1.029 \cos(120\pi t - 0.3605) \text{ (amps)} \quad (\text{argument of cosine function} \sim \text{radians})$$

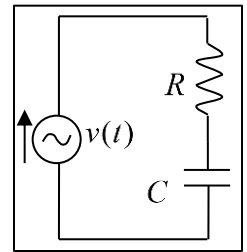
Note: We use only the **phase angle** of $v(t)$ and not “ ωt ” when we write the complex form. We must include the “ ωt ” when we write $i(t)$.

Example #5

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RC series circuit with $R = 100$ (ohms) and $C = 20$ (μf). The impedance of the circuit is

$$\boxed{Z = Z_R + Z_C} \text{ and the frequency is } \boxed{\omega = 120\pi \text{ (rad/s)}}.$$



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, the impedance is $\boxed{Z = Z_R + Z_C \approx 100 - j132.6 \text{ (ohms)}}$

or

$$\boxed{Z \approx 166.1e^{j(-52.98^\circ)} \approx 166.1 \angle (-52.98^\circ) \text{ (ohms)}}$$

So, the complex current is

$$\boxed{I = V/Z \approx 110e^{j(0^\circ)} / 166.1e^{j(-52.98^\circ)} \approx 0.6623e^{j(52.98^\circ)} \text{ (amps)}}$$

and

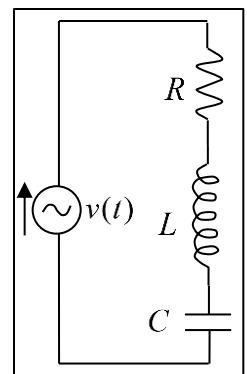
$$\boxed{i(t) \approx 0.6623 \cos(120\pi t + 0.9246) \text{ (amps)}} \text{ (argument of cosine function } \sim \text{ radians)}$$

Example #6

Given:

A voltage $v(t) = 110 \cos(120\pi t)$ is applied to an RLC series circuit with $R = 100$ (ohms), $L = 100$ (mh), and $C = 20$ (μf). The impedance of the circuit is

$$\boxed{Z = Z_R + Z_L + Z_C} \text{ and the frequency is } \boxed{\omega = 120\pi \text{ (rad/s)}}.$$



Find: Find the complex current I and the time-based current $i(t)$.

Solution:

From previous results, we have $\boxed{Z = Z_R + Z_L + Z_C \approx 100 - j94.9 \text{ (ohms)}}$

or

$$\boxed{Z \approx 137.9e^{j(-43.5^\circ)} \approx 137.9 \angle (-43.5^\circ) \text{ (ohms)}}$$

So, the complex current is

$$\boxed{I = V/Z \approx 110e^{j(0^\circ)} / 137.9e^{j(-43.5^\circ)} \approx 0.7977e^{j(43.5^\circ)} \text{ (amps)}}$$

and

$$i(t) = 0.7977 \cos(120\pi t + 0.7592) \text{ (amps)} \quad (\text{argument of cosine function} \sim \text{radians})$$

The graph below shows a plot of three cycles of the voltage $v(t)$ and current $i(t)$ for the RLC circuit. Note the current is multiplied by 100 to make it visible on the graph.

