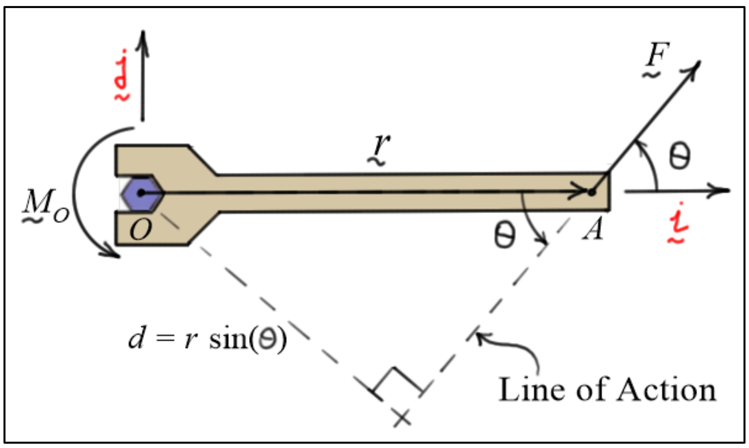
Elementary Statics
Moments of Forces and the Cross Product

Moment of a Force – Torque

- The ***moment*** (or ***torque***) of a force about a point O is defined as the ***magnitude of the force*** $\left(|\underset{\sim}{F}|\right)$ multiplied by the ***perpendicular distance*** from the ***point*** to the ***line of action*** of the force $\left(d = r\sin(\theta)\right)$. $\boxed{|\underset{\sim}{M}_O| = |\underset{\sim}{F}|\, r\sin(\theta)}$
- The ***direction*** of the moment is defined by the ***right-hand-rule***. Let the fingers of your right hand show the direction of the ***circulation*** of $\underset{\sim}{F}$ around O, and your ***thumb*** shows the direction of the moment. $\boxed{\underset{\sim}{M}_O = |\underset{\sim}{F}|\, r\sin(\theta)\ \underset{\sim}{k}}$.

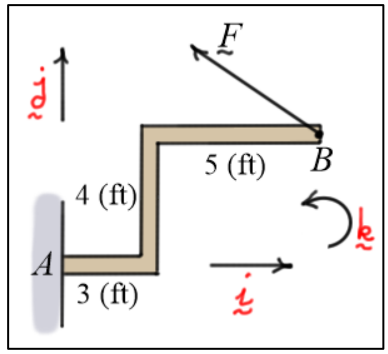


- The moment of $\underset{\sim}{F}$ about O can also be calculated by first ***breaking*** the force into ***components***, and then ***summing*** the moments of the individual components.
- As an example, consider the force $\underset{\sim}{F}$ shown in the diagram. The ***line of action*** of the ***X-component*** of $\underset{\sim}{F}$ passes through O and, hence, has ***no moment*** about O. The ***line of action*** of the ***Y-component*** is ***perpendicular*** to the position vector $\underset{\sim}{r}$. So, the moment of $\underset{\sim}{F}$ can be calculated as

$$\boxed{\underset{\sim}{M}_O = \left[\underbrace{\left(|\underset{\sim}{F}|\cos(\theta)\right)}_{X\text{ - component}} \cdot 0\right]\underset{\sim}{k} + \left[\underbrace{\left(|\underset{\sim}{F}|\sin(\theta)\right)}_{Y\text{ - component}} \cdot r\right]\underset{\sim}{k} = \left(|\underset{\sim}{F}|\sin(\theta)\, r\right)\underset{\sim}{k}}$$

Example l:

Given: Force $\underset{\sim}{F} = -300\,\underset{\sim}{i} + 100\,\underset{\sim}{j}$ (lb) is applied at point *B*.



Find: $\underset{\sim}{M}_A$ the moment of $\underset{\sim}{F}$ about point *A*.

Solution:

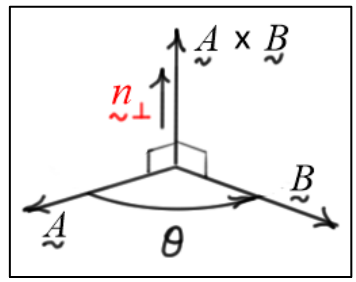
$$\underset{\sim}{M}_A = \left[(4 \cdot 300) + (8 \cdot 100)\right]\underset{\sim}{k} = 2000\,\underset{\sim}{k} \text{ (ft-lb)}$$

The Cross Product

Geometric Definition

- The ***cross*** product of two vectors is defines as

$$\boxed{\underset{\sim}{A} \times \underset{\sim}{B} = \left(\left|\underset{\sim}{A}\right| \left|\underset{\sim}{B}\right| \sin(\theta)\right) \underset{\sim}{n}_{\perp}}$$



- Here, $\underset{\sim}{n}_{\perp}$ is a ***unit vector perpendicular*** to the plane formed by the two vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$.
- The sense of $\underset{\sim}{n}_{\perp}$ is defined by the ***right-hand-rule***, that is, the ***right thumb*** points in the direction of $\underset{\sim}{n}_{\perp}$ when the ***fingers*** of the right hand point from $\underset{\sim}{A}$ to $\underset{\sim}{B}$.

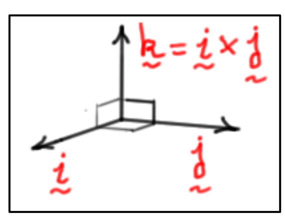
Properties of the Cross Product

- Product is ***not commutative***: $\underset{\sim}{A} \times \underset{\sim}{B} = -\left(\underset{\sim}{B} \times \underset{\sim}{A}\right)$
- Product is ***distributive*** over ***addition***: $\underset{\sim}{A} \times \left(\underset{\sim}{B} + \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \times \underset{\sim}{B}\right) + \left(\underset{\sim}{A} \times \underset{\sim}{C}\right)$
- Multiplication by a ***scalar*** α: $\alpha\left(\underset{\sim}{A} \times \underset{\sim}{B}\right) = \left(\alpha \underset{\sim}{A}\right) \times \left(\alpha \underset{\sim}{B}\right)$

Calculation

- Cross products of the unit vectors of a right-handed set of three mutually perpendicular unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k} = \underset{\sim}{i} \times \underset{\sim}{j}$ produce the following results.

  - $\underset{\sim}{i} \times \underset{\sim}{i} = \underset{\sim}{0}$ $\quad$ $\underset{\sim}{j} \times \underset{\sim}{j} = \underset{\sim}{0}$ $\quad$ $\underset{\sim}{k} \times \underset{\sim}{k} = \underset{\sim}{0}$
  - $\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$ $\quad$ $\underset{\sim}{j} \times \underset{\sim}{k} = \underset{\sim}{i}$ $\quad$ $\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j}$
  - $\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k}$ $\quad$ $\underset{\sim}{k} \times \underset{\sim}{j} = -\underset{\sim}{i}$ $\quad$ $\underset{\sim}{i} \times \underset{\sim}{k} = -\underset{\sim}{j}$



- Using the ***properties*** of the cross product and the results given above for the cross products of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, the cross product of two vectors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ can be shown to produce the following result.

$$\boxed{\begin{aligned}\underset{\sim}{A}\times\underset{\sim}{B} &= \left(a_x\,\underset{\sim}{i}+a_y\,\underset{\sim}{j}+a_z\,\underset{\sim}{k}\right)\times\left(b_x\,\underset{\sim}{i}+b_y\,\underset{\sim}{j}+b_z\,\underset{\sim}{k}\right)\\ &= \left(a_yb_z-a_zb_y\right)\underset{\sim}{i}-\left(a_xb_z-a_zb_x\right)\underset{\sim}{j}+\left(a_xb_y-a_yb_x\right)\underset{\sim}{k}\end{aligned}}$$

- The ***cross product*** of any two vectors is ***zero*** if they are ***parallel***.
- The result shown in the boxed equation above can be calculated using the following ***matrix determinant form***. The determinant is expanded using the ***cofactors*** of the unit vectors which are listed in the first row.

$$\underset{\sim}{A}\times\underset{\sim}{B}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\overset{\left(+\underset{\sim}{i}\right)}{\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}}+\overset{\left(-\underset{\sim}{j}\right)}{\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}}+\overset{\left(+\underset{\sim}{k}\right)}{\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}}$$

$$=\left(a_yb_z-a_zb_y\right)\underset{\sim}{i}-\left(a_xb_z-a_zb_x\right)\underset{\sim}{j}+\left(a_xb_y-a_yb_x\right)\underset{\sim}{k}$$

Moment of a Force – Using the Cross Product

The ***moment*** of a force about a point O can be calculated using the ***cross product***

$$\boxed{\underset{\sim}{M}_O=\underset{\sim}{r}\times\underset{\sim}{F}}$$

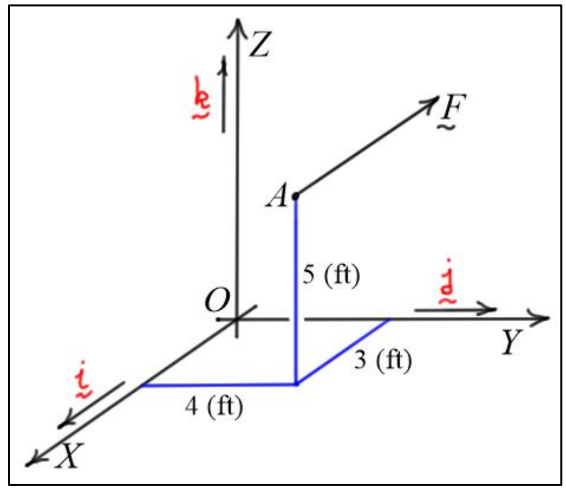
Here, $\underset{\sim}{r}$ is a ***position vector*** from O to ***any point*** on the ***line of action*** of $\underset{\sim}{F}$.

Example 2:

Given: Force $\underset{\sim}{F} = -100\underset{\sim}{i} + 50\underset{\sim}{j} + 200\underset{\sim}{k}$ (lb),

Point $A:(3,4,5)$ (ft)

Find: $\underset{\sim}{M}_O$ the moment of $\underset{\sim}{F}$ about O



Solution:

$$\underset{\sim}{M}_O = \underset{\sim}{r}_{A/O} \times \underset{\sim}{F}$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 4 & 5 \\ -100 & 50 & 200 \end{vmatrix}$$

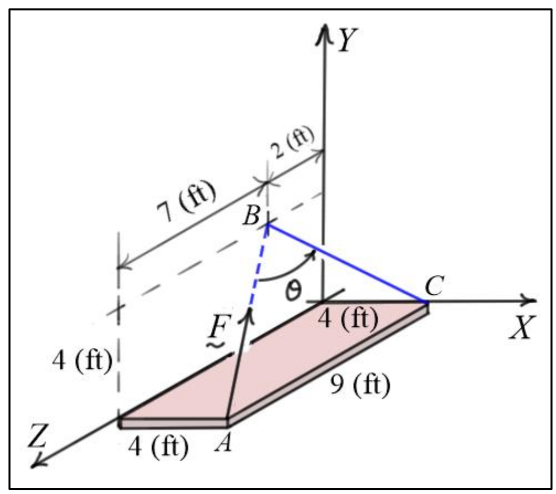
$$= (800-250)\underset{\sim}{i} - (600+500)\underset{\sim}{j} + (150+400)\underset{\sim}{k}$$

$$= 550\underset{\sim}{i} - 1100\underset{\sim}{j} + 550\underset{\sim}{k} \text{ (ft-lb)}$$

Example #3:

Given: Force $\underset{\sim}{F}$ is applied to the rectangular plate as shown. $|\underset{\sim}{F}| = 108$ (lb)

Find: $\underset{\sim}{M}_C$ the moment of $\underset{\sim}{F}$ about point C.



Solution:

The unit vector pointing from A to B can be calculated using the dimensions shown in the figure as follows.

$$\boxed{\underset{\sim}{u}_{AB} = \left(-4\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}\right) \Big/ \sqrt{4^2+4^2+7^2} = -\tfrac{4}{9}\underset{\sim}{i} + \tfrac{4}{9}\underset{\sim}{j} - \tfrac{7}{9}\underset{\sim}{k}}$$

The force $\underset{\sim}{F}$ can then be written as

$$\boxed{\underset{\sim}{F} = 108\underset{\sim}{u}_{AB} = 108\left(-\tfrac{4}{9}\underset{\sim}{i} + \tfrac{4}{9}\underset{\sim}{j} - \tfrac{7}{9}\underset{\sim}{k}\right) = -48\underset{\sim}{i} + 48\underset{\sim}{j} - 84\underset{\sim}{k}}$$

The moment of $\underset{\sim}{F}$ about point C can be calculated as follows.

$$\underset{\sim}{M}_C = \underset{\sim}{r}_{A/C} \times \underset{\sim}{F} = 9\underset{\sim}{k} \times \left(-48\underset{\sim}{i} + 48\underset{\sim}{j} - 84\underset{\sim}{k}\right) \Rightarrow \boxed{\underset{\sim}{M}_C = -432\underset{\sim}{i} - 432\underset{\sim}{j}}$$

Check:

Because we can pick any point on the line of action of the force, the moment of $\underset{\sim}{F}$ about point C can also be calculated as follows.

$$\underset{\sim}{M}_C = \underset{\sim}{r}_{B/C} \times \underset{\sim}{F} = \left(-4\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}\right) \times \left(-48\underset{\sim}{i} + 48\underset{\sim}{j} - 84\underset{\sim}{k}\right)$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & 4 & 2 \\ -48 & 48 & -84 \end{vmatrix} = \left(-4(84) - 2(48)\right)\underset{\sim}{i} - \left(4(84) + 2(48)\right)\underset{\sim}{j} + \left(-4(48) + 4(48)\right)\underset{\sim}{k}$$

$$\Rightarrow \boxed{\underset{\sim}{M}_C = -432\underset{\sim}{i} - 432\underset{\sim}{j}} \quad \text{…same result}$$

Resultant Moment

As we did with forces, we can define a ***resultant moment*** about a point *O*. This is defined as the ***sum*** of the moments of ***all*** the ***forces*** about point *O*. For a system of *N* forces,

$$\boxed{\left(\underset{\sim}{M}_O\right)_R = \sum_{i=1}^{N} \left(\underset{\sim}{M}_O\right)_i = \sum_{i=1}^{N} \left(\underset{\sim}{r}_i \times \underset{\sim}{F}_i\right)}$$

Here, $\underset{\sim}{r}_i \; (i = 1, \ldots, N)$ are vectors from point *O* to the lines of actions of each of the forces.