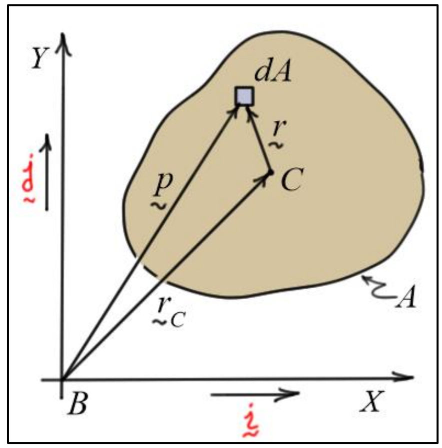
Elementary Statics
Centroids of Areas



Definition of the Centroid of an Area

- The figure depicts an ***area*** A in the XY plane. ***Centroid*** C of the area is defined as the point where

$$\boxed{\int_A \underset{\sim}{r}\, dA = 0} \quad \Rightarrow \begin{cases} \int_A x'\, dA = 0 \\ \int_A y'\, dA = 0 \end{cases}$$

Here, the ***position vector*** of the ***area element*** dA relative to C is $\boxed{\underset{\sim}{r} = x'\underset{\sim}{i} + y'\underset{\sim}{j}}$.

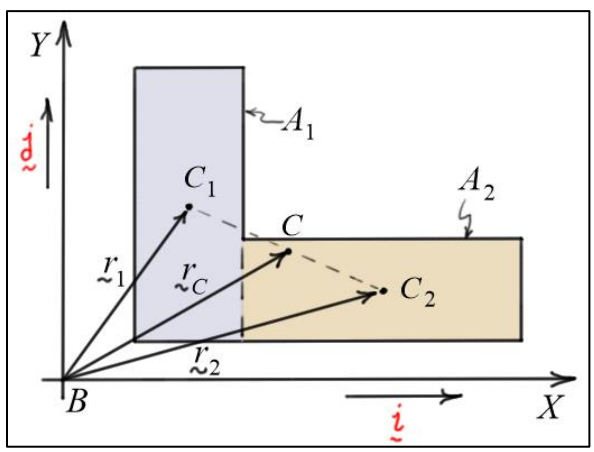
- A more ***practical definition*** of the ***location*** of C can be developed as follows. First, choose some ***arbitrary location*** B as a ***reference point***, then calculate

$$\int_A \underset{\sim}{p}\, dA = \int_A \left(\underset{\sim}{r}_C + \underset{\sim}{r}\right) dA = \left(\int_A dA\right)\underset{\sim}{r}_C + \underbrace{\int_A \underset{\sim}{r}\, dA}_{\text{zero}} = A\underset{\sim}{r}_C \quad \Rightarrow \boxed{\underset{\sim}{r}_C = \frac{1}{A}\int_A \underset{\sim}{p}\, dA = \frac{\int_A \underset{\sim}{p}\, dA}{\int_A dA}}$$

- This result can be expressed as two scalar equations for $\bar{x}$ and $\bar{y}$, the X and Y ***coordinates*** of C relative to B. Here, $\boxed{\underset{\sim}{p} = x\underset{\sim}{i} + y\underset{\sim}{j}}$.

$$\boxed{\underset{\sim}{r}_C = \frac{1}{A}\int_A \underset{\sim}{p}\, dA} \quad \Rightarrow \begin{cases} \bar{x} = \frac{1}{A}\int_A x\, dA \\ \bar{y} = \frac{1}{A}\int_A y\, dA \end{cases}$$

Centroids of Composite Areas

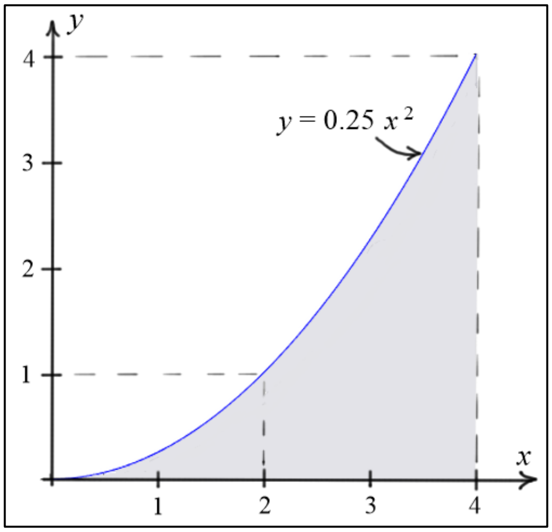


- To determine the ***centroid*** of a ***composite area***, first break down the area into a set of areas A_i, all having ***common geometric shapes***.
- Then use the ***tables*** to determine the locations of the centroids of these common shapes relative to some chosen (arbitrary) location B.

- Finally, ***compute*** the location of the centroid of the composite shape as follows

$$\boxed{\underset{\sim}{r}_C = \frac{\sum_i A_i \, \underset{\sim}{r}_i}{\sum_i A_i}} \quad \Rightarrow \begin{cases} \bar{x} = \dfrac{\sum A_i \, \bar{x}_i}{\sum A_i} \\[2ex] \bar{y} = \dfrac{\sum A_i \, \bar{y}_i}{\sum A_i} \end{cases}$$

Example #1: (direct integration)



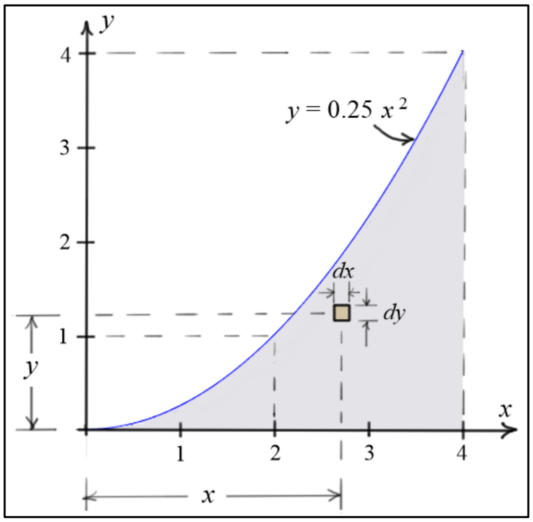
Given: shaded area under the function $y = 0.25x^2$ from $x = 0$ to $x = 4$. Coordinates in inches.

Find: $\bar{x}$ and $\bar{y}$ the coordinates of the centroid of the shaded area.

Solution:

Approach #1:

Use a differential element $dA = dx \times dy$ with centroid located at $(\bar{x}_e, \bar{y}_e) = (x, y)$.



Area:

$$\boxed{A = \int_0^4 \left(\int_0^{0.25x^2} dy \right) dx = \int_0^4 0.25x^2 dx = \tfrac{1}{12} x^3 \Big|_0^4 = \tfrac{16}{3} \approx 5.33 \text{ (in}^2\text{)}}$$

$\bar{x}$:

$$\bar{x} = \frac{1}{A} \int_0^4 \left(\int_0^{0.25x^2} x \, dy \right) dx = \frac{3}{16} \int_0^4 x \left(\int_0^{0.25x^2} dy \right) dx = \frac{3}{16} \int_0^4 \tfrac{1}{4} x^3 dx$$

$$= \tfrac{3}{16} \left(\tfrac{1}{16} x^4 \right) \Big|_0^4 = \tfrac{3}{16^2} \left(4^4 \right) \quad \Rightarrow \boxed{\bar{x} = 3 \text{ (in)}}$$

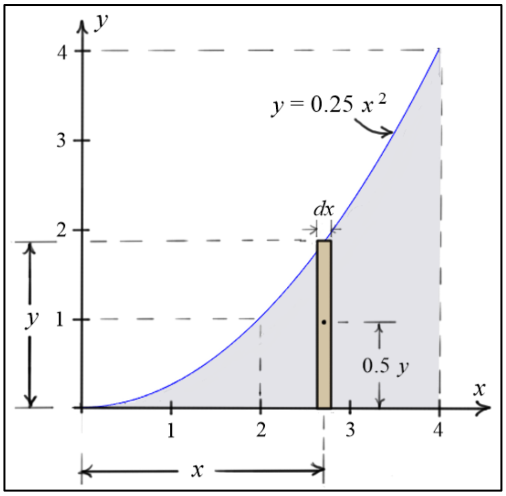
$\bar{y}$:

$$\bar{y} = \frac{1}{A} \int_0^4 \left(\int_0^{0.25x^2} y \, dy \right) dx = \frac{1}{A} \int_0^4 \left[\tfrac{1}{2} y^2 \right]_0^{0.25x^2} dx = \frac{3}{16} \int_0^4 \tfrac{1}{2} \left(\tfrac{1}{4} x^2 \right)^2 dx = \tfrac{3}{16} \left(\tfrac{1}{32} \right) \left[\tfrac{1}{5} x^5 \right]_0^4 = \tfrac{3}{2560} \left(4^5 \right)$$

$$\Rightarrow \boxed{\bar{y} = \tfrac{6}{5} = 1.2 \text{ (in)}}$$

Approach #2:

Use a vertical strip differential element $dA = y\,dx$ with centroid at $(\bar{x}_e, \bar{y}_e) = (x, \frac{1}{2}y)$.



Area:

$$A = \int dA$$
$$= \int_0^4 y\,dx = \int_0^4 \tfrac{1}{4}x^2\,dx = \tfrac{1}{4}\left(\tfrac{1}{3}x^3\right)_0^4 = \tfrac{1}{12}\left(4^3\right) = \tfrac{16}{3} \approx 5.33\ (\text{in}^2)$$

$\bar{x}$:

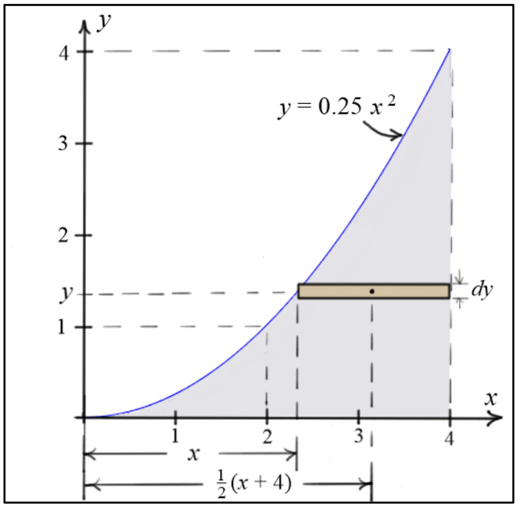
$$\bar{x} = \tfrac{1}{A}\int \bar{x}_e\,dA = \tfrac{3}{16}\int_0^4 x\,y\,dx = \tfrac{3}{16}\int_0^4 \tfrac{1}{4}x^3\,dx = \tfrac{3}{64}\left(\tfrac{1}{4}x^4\right)_0^4 = \tfrac{3}{256}\left(4^4\right) \quad \Rightarrow \boxed{\bar{x} = 3\ (\text{in})}$$

$\bar{y}$:

$$\bar{y} = \tfrac{1}{A}\int \bar{y}_e\,dA = \tfrac{3}{16}\int_0^4 \left(\tfrac{1}{2}y\right)y\,dx = \tfrac{3}{16}\int_0^4 \tfrac{1}{2}\left(\tfrac{1}{4}x^2\right)^2 dx = \tfrac{3}{512}\left(\tfrac{1}{5}x^5\right)_0^4 = \tfrac{3}{2560}\left(4^5\right) \quad \Rightarrow \boxed{\bar{y} = \tfrac{6}{5} = 1.2\ (\text{in})}$$

Approach #2:

Use a horizontal strip differential element $dA = (4-x)\,dy$ with centroid at $(\bar{x}_e, \bar{y}_e) = (\frac{1}{2}(x+4), y)$.



Area:

$$A = \int dA$$
$$= \int_0^4 (4-x)\,dy = \int_0^4 \left(4-\sqrt{4y}\right)dy = \int_0^4 \left(4-2y^{\frac{1}{2}}\right)dy$$
$$= \left(4y - \tfrac{4}{3}y^{\frac{3}{2}}\right)_0^4 = \left(16 - \tfrac{4}{3}(8)\right) = \tfrac{48-32}{3} = \tfrac{16}{3} \approx 5.33\ (\text{in}^2)$$

$\bar{x}$:

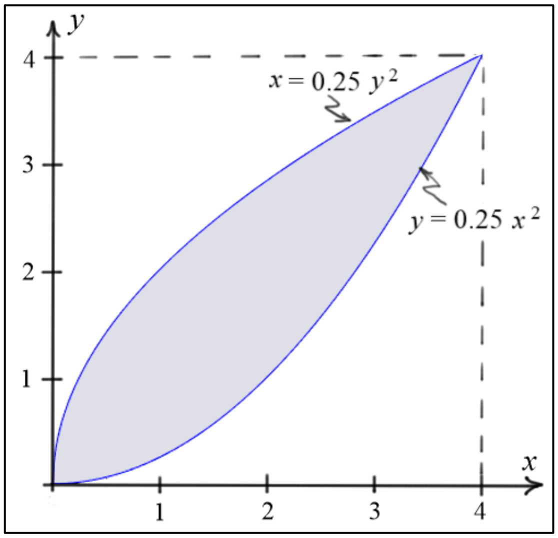
$$\bar{x} = \tfrac{1}{A}\int \bar{x}_e\,dA = \tfrac{3}{16}\int_0^4 \tfrac{1}{2}(x+4)(4-x)\,dy = \tfrac{3}{16}\int_0^4 \tfrac{1}{2}\left(16-x^2\right)dy = \tfrac{3}{32}\int_0^4 (16-4y)\,dy$$
$$= \tfrac{3}{32}\left(16y-2y^2\right)_0^4 = \tfrac{3}{32}(64-32) \quad \Rightarrow \boxed{\bar{x} = 3\ (\text{in})}$$

$\bar{y}$:

$$\bar{y} = \tfrac{1}{A}\int \bar{y}_e\,dA = \tfrac{3}{16}\int_0^4 y(4-x)\,dy = \tfrac{3}{16}\int_0^4 y\left(4-2y^{\frac{1}{2}}\right)dy = \tfrac{3}{16}\int_0^4 \left(4y-2y^{\frac{3}{2}}\right)dy$$
$$= \tfrac{3}{16}\left(2y^2 - \tfrac{4}{5}y^{\frac{5}{2}}\right)_0^4 = \tfrac{3}{16}\left(32-\tfrac{4}{5}(32)\right) = \tfrac{3}{16}\left(\tfrac{32}{5}\right) \quad \Rightarrow \boxed{\bar{y} = \tfrac{6}{5} = 1.2\ (\text{in})}$$

Observations:

- The solution provided in Approach #1 follows very closely the solution provided in Approach #2. The reason for this is the y integration was done first. In that integration, results for the vertical strip differential element are generated. The second integration follows closely the single integration followed in Approach #2.
- If in Approach #1 the x integration was completed first, the results for the horizontal strip differential element would have been generated in that first integration. The second integration would follow the details provided by the single integration in Approach #3.
- Clearly the choice of differential elements does not alter the results, but it can alter the level of complexity of the details



Example #2: (direct integration)

Given: shaded area between the two functions $y = 0.25x^2$ and $x = 0.25y^2$ from $x = 0$ to $x = 4$. Coordinates in inches.

Find: $\bar{x}$ and $\bar{y}$ the coordinates of the centroid of the shaded area.

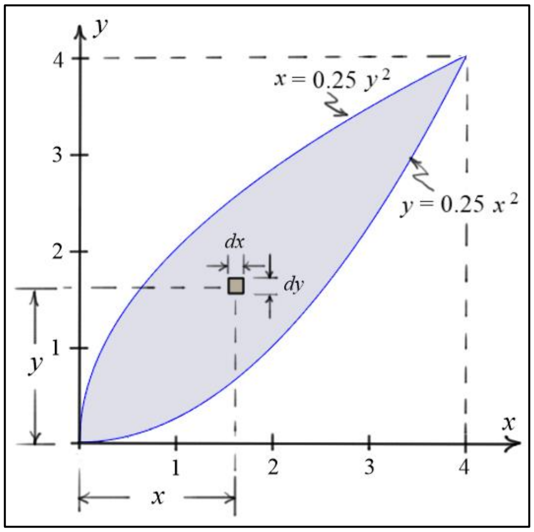
Solution:

Area:

$$A = \int dA = \int_0^4 \int_{0.25x^2}^{2\sqrt{x}} dy\,dx = \int_0^4 \left(y\right)_{0.25x^2}^{2\sqrt{x}} dx = \int_0^4 \left(2x^{\frac{1}{2}} - \tfrac{1}{4}x^2\right) dx$$

$$= \left(\tfrac{4}{3}x^{\frac{3}{2}} - \tfrac{1}{12}x^3\right)_0^4 = \tfrac{4}{3}\left(4^{\frac{3}{2}}\right) - \tfrac{4^3}{12} = \tfrac{32}{3} - \tfrac{16}{3}$$

$$\Rightarrow \boxed{A = \tfrac{16}{3} \approx 5.33} \;\; (\text{in}^2)$$



$\bar{x}$:

$$\bar{x} = \tfrac{1}{A}\int x\,dA = \tfrac{3}{16}\int_0^4 \int_{0.25x^2}^{2\sqrt{x}} x\,dy\,dx = \tfrac{3}{16}\int_0^4 x\left(y\right)_{0.25x^2}^{2\sqrt{x}} dx$$

$$= \tfrac{3}{16}\int_0^4 \left(2x^{\frac{3}{2}} - \tfrac{1}{4}x^3\right) dx = \tfrac{3}{16}\left(\tfrac{4}{5}x^{\frac{5}{2}} - \tfrac{1}{16}x^4\right)_0^4 = \tfrac{3}{16}\left(\tfrac{4}{5}\left(4^{\frac{5}{2}}\right) - \tfrac{4^4}{16}\right)$$

$$= \tfrac{3}{16}\left(\tfrac{128}{5} - 16\right) = \tfrac{3}{16}\left(\tfrac{128-80}{5}\right) = \tfrac{3}{16}\left(\tfrac{48}{5}\right) \quad \Rightarrow \boxed{\bar{x} = \tfrac{9}{5} \approx 1.8} \;\; (\text{in})$$

$\bar{y}$:

$$\bar{y}=\tfrac{1}{A}\int y\,dA=\tfrac{3}{16}\int_0^4\int_{0.25x^2}^{2\sqrt{x}} y\,dy\,dx=\tfrac{3}{16}\int_0^4\left(\tfrac{1}{2}y^2\right)_{0.25x^2}^{2\sqrt{x}}dx=\tfrac{3}{16}\int_0^4\left(2x-\tfrac{1}{32}x^4\right)dx$$

$$=\tfrac{3}{16}\left(x^2-\tfrac{1}{160}x^5\right)_0^4=\tfrac{3}{16}\left(4^2-\tfrac{1}{160}\left(4^5\right)\right)=\tfrac{3}{16}\left(16-\tfrac{32}{5}\right)=\tfrac{3}{16}\left(\tfrac{80-32}{5}\right)\tfrac{3}{16}\left(\tfrac{48}{5}\right)$$

$$\Rightarrow \boxed{\bar{y}=\tfrac{9}{5}\approx 1.8} \text{ (in)}$$

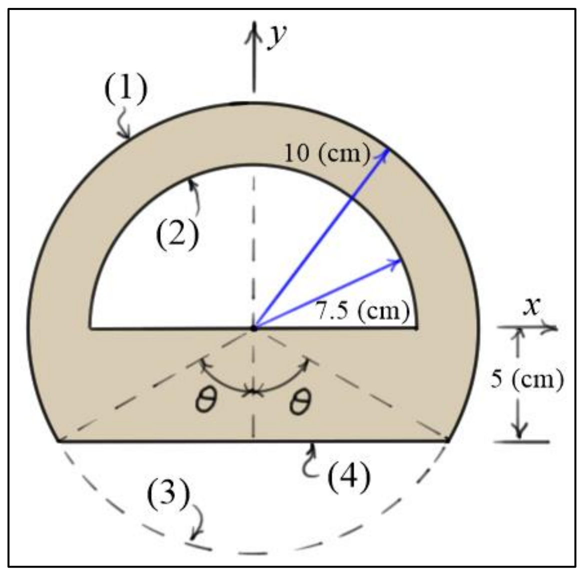
Observations:

- The shaded area is symmetrical about the line $y=x$ so the centroid lies along that line making $\bar{y}=\bar{x}$. Our results are consistent with this observation.
- Because the *y* integration was completed first, the calculations above are like those completed when using a vertical slice differential element. You may want to set up the integral using a vertical slice differential element to see the similarity.
- You may also want to try the calculations by completing the *x* integration first and compare the second integral to the integral used with a horizontal slice differential element.

<u>Example #3</u>: (composite shapes)

Given: Composite shape shown.

Find: Centroid of the composite shape.



Solution:

- Due to the symmetry of the shape about the *yz*-plane, $\bar{x}=0$.
- The shape can be thought of as a composite of four areas of common shapes – circular, semicircular, circular sector, and triangular.
- Missing areas have negative values.

Using a table of centroids of common shapes:

Circular area (1):

$$A_1=\pi r^2=\pi\left(10^2\right)=100\pi \text{ (cm}^2\text{)}$$

$$\bar{x}_1=\bar{y}_1=0$$

Semicircular area (2): (missing area)

$$A_2 = -\tfrac{1}{2}\pi r^2 = -\tfrac{1}{2}\pi\left(7.5^2\right) = -\frac{225\pi}{8}\ \text{(cm}^2\text{)}$$

$$\bar{x}_2 = 0 \qquad \bar{y}_2 = \frac{4r}{3\pi} = \frac{4(7.5)}{3\pi} = \frac{10}{\pi}\ \text{(cm)}$$

Circular sector area (3): (missing area)

sector half angle: $\theta = \cos^{-1}\left(\tfrac{5}{10}\right) = 60\ \text{(deg)} = \tfrac{\pi}{3}\ \text{(rad))}$

$$A_3 = -\left(\frac{\pi}{3}\right)r^2 = -\left(\frac{\pi}{3}\right)10^2 = -\frac{100\pi}{3}\ \text{(cm}^2\text{)}; \quad \bar{x}_3 = 0$$

$$\bar{y}_3 = -\frac{2}{3}\left(\frac{r\sin(\theta)}{\theta}\right) = -\frac{2}{3}\left(\frac{r\sin(\frac{\pi}{3})}{\frac{\pi}{3}}\right) = -\frac{2}{3}\left(\frac{10\frac{\sqrt{3}}{2}}{\frac{\pi}{3}}\right) = \frac{-10\sqrt{3}}{\pi}\ \text{(cm)}$$

Triangular area (4):

$$A_4 = \tfrac{1}{2}bh = \tfrac{1}{2}\left(2(10\sin(\theta)\right)5 = 25\sqrt{3}\ \text{(cm}^2\text{)}$$

$$\bar{x}_4 = 0 \qquad \bar{y}_4 = -\tfrac{2}{3}h = \frac{-10}{3}\ \text{(cm)}$$

Composite Area:

$$A = A_1 + A_2 + A_3 + A_4 = \left(100\pi\right) + \left(-\frac{225\pi}{8}\right) + \left(-\frac{100\pi}{3}\right) + \left(25\sqrt{3}\right) = \frac{925\pi}{24} + \left(25\sqrt{3}\right)$$

$$\Rightarrow \boxed{A \approx 164.383}\ \text{(cm}^2\text{)}$$

$$\sum_{i=1}^{4} A_i\,\bar{y}_i = 100\pi\left(0\right) + \left(-\frac{225\not\pi}{8}\right)\left(\frac{10}{\not\pi}\right) + \left(-\frac{100\not\pi}{3}\right)\left(\frac{-10\sqrt{3}}{\not\pi}\right) + \left(25\sqrt{3}\right)\left(\frac{-10}{3}\right)$$

$$= -\frac{2250}{8} + \frac{1000\sqrt{3}}{3} - \frac{250\sqrt{3}}{3} = -\frac{2250}{8} + \frac{750\sqrt{3}}{3}$$

$$\Rightarrow \boxed{\sum_{i=1}^{4} A_i\,\bar{y}_i \approx 151.763\ \text{(cm}^3\text{)}}$$

$$\boxed{\bar{y} = \frac{\sum A_i\,\bar{y}_i}{\sum A_i} \approx \frac{151.763}{164.383} \approx 0.923\ \text{(cm)} \approx 9.23\ \text{(mm)}}$$