

Elementary Engineering Mathematics

Exponential and Natural Logarithm Functions

The exponential function is defined as: $f(x) = e^x$ ($e = 2.71828\dots$). This function is useful for describing **growth** or **decay rates**. We will see in later notes how it is used to describe the motion of a mass-spring-damper system. Figures 1 and 2 show exponential functions having various growth and decay rates. The **larger** the exponent (in absolute value), the **higher** the growth or decay rate.

As defined, the exponential function is related to the natural logarithm function ($\log_e(x)$). In fact, the exponential function and the natural logarithm functions are **inverses** of each other. As a result,

$$\log_e(e^x) = \ln(e^x) = x \quad \text{and} \quad e^{\log_e(x)} = e^{\ln(x)} = x \quad (1)$$

Figure 3 shows a plot of the exponential and natural logarithm functions. They are mirror images about the line $y = x$.

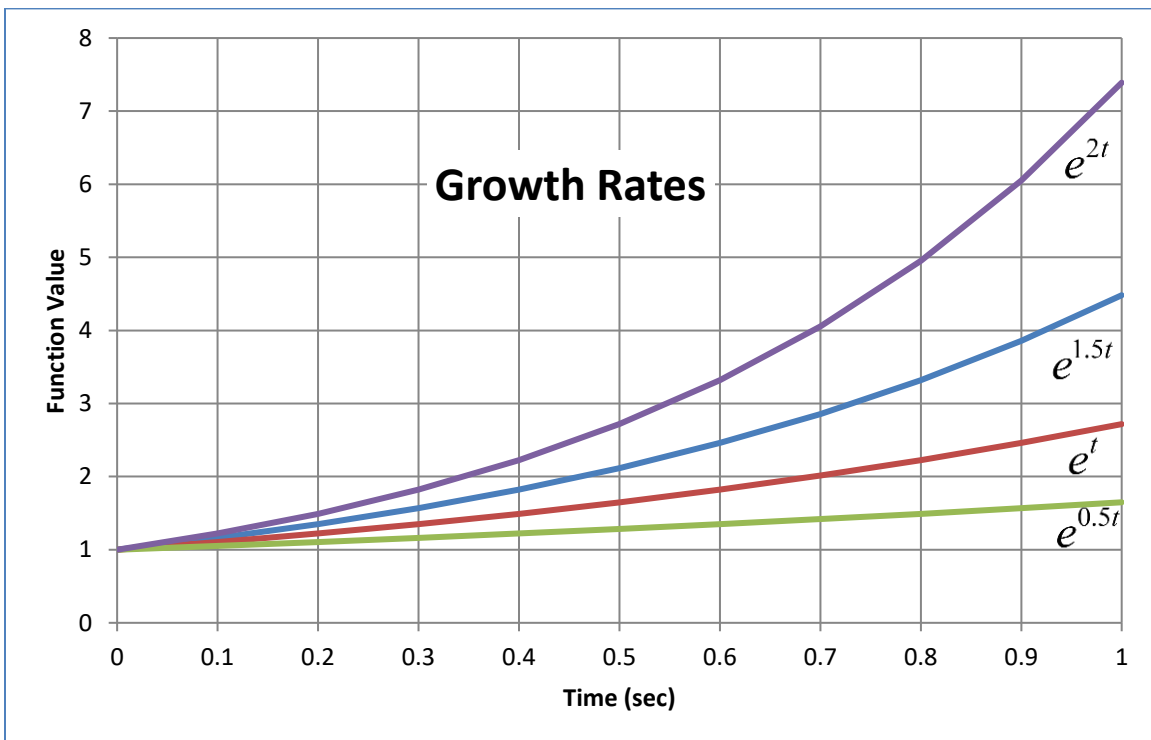


Figure 1. Exponential Growth Rates

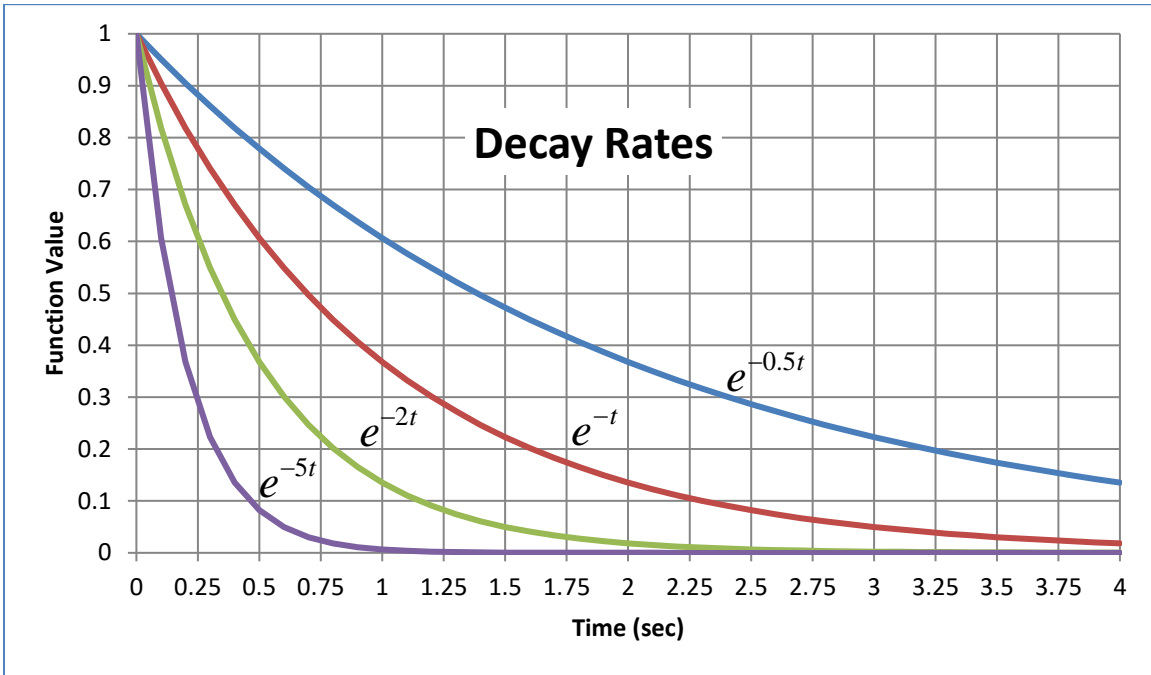


Figure 2. Exponential Decay Rates

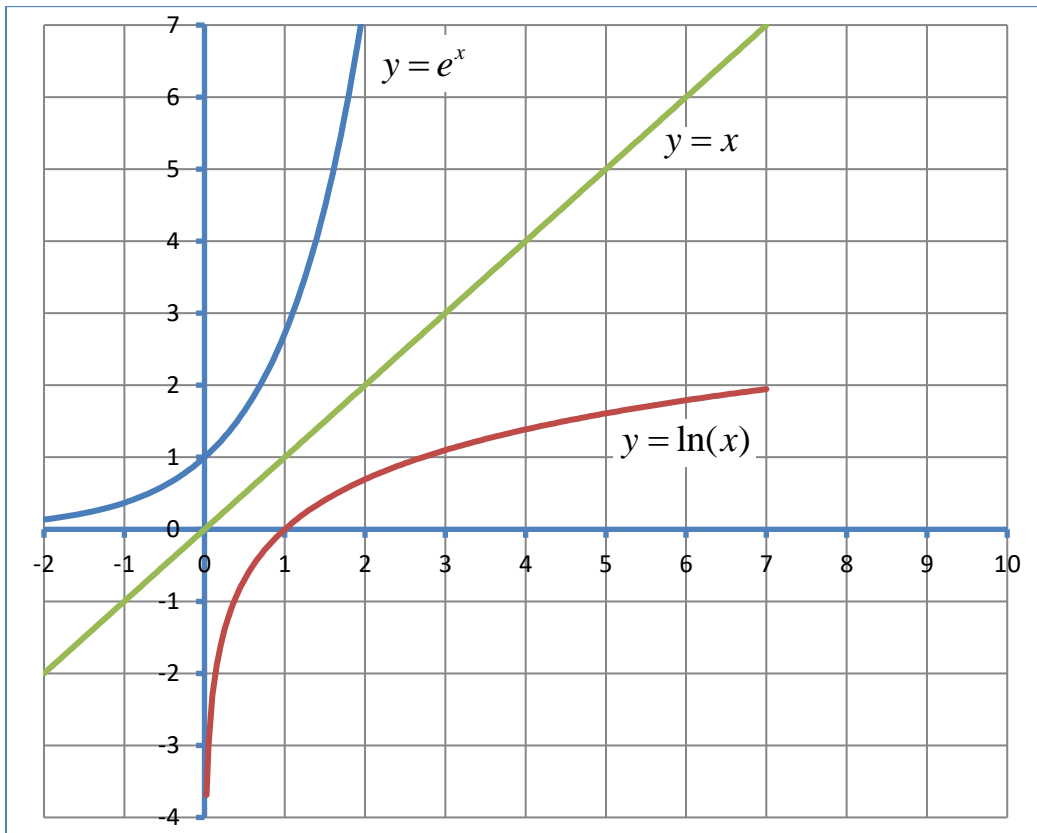
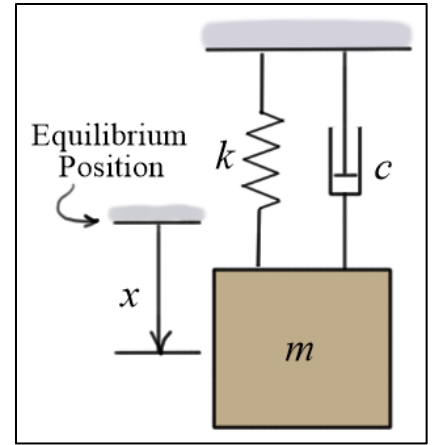


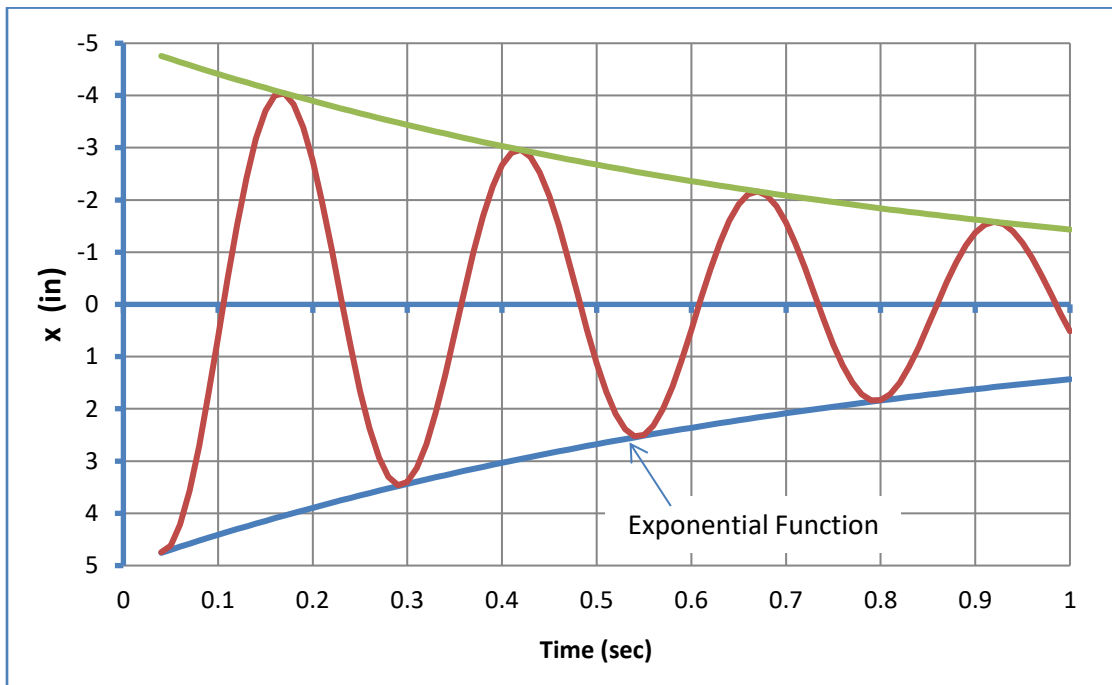
Figure 3. Exponential and Logarithm Functions

Example #1: Displacement of a spring-mass-damper system

Given: The displacement of a mass-spring-damper system is shown by the red line in the plot below. The displacement is bounded by an exponential function shown in blue. The displacement was measured at its four bottom-most positions during the one-second period to be



Time, t (sec)	Displacement, x (in)
0.04	4.74
0.29	3.46
0.54	2.52
0.79	1.83



Find: Using the data provided, estimate the decay rate α of the oscillations. Assume the displacement function is bounded by an exponential function $x = Ae^{\alpha t}$.

Solution:

Given that all four points lie on the same exponential function, we can estimate the decay rate by using any pair of data values as follows

$$\frac{x_2}{x_1} = \frac{Ae^{\alpha t_2}}{Ae^{\alpha t_1}} = e^{\alpha(t_2 - t_1)} \Rightarrow \ln\left(\frac{x_2}{x_1}\right) = \ln\left(e^{\alpha(t_2 - t_1)}\right) = \alpha(t_2 - t_1)$$

or

$$\alpha = \frac{\ln(x_2/x_1)}{t_2 - t_1} = \frac{\ln(x_2) - \ln(x_1)}{t_2 - t_1} \quad (2)$$

Using Eq. (2), we can compute estimates of the decay rate.

Time, t (sec)	Displacement, x (in)	Decay Rate, α
0.04	4.74	
0.29	3.46	-1.259
0.54	2.52	-1.268
0.79	1.83	-1.28
Average		-1.269

Example #2: Transient response of a direct current RC circuit

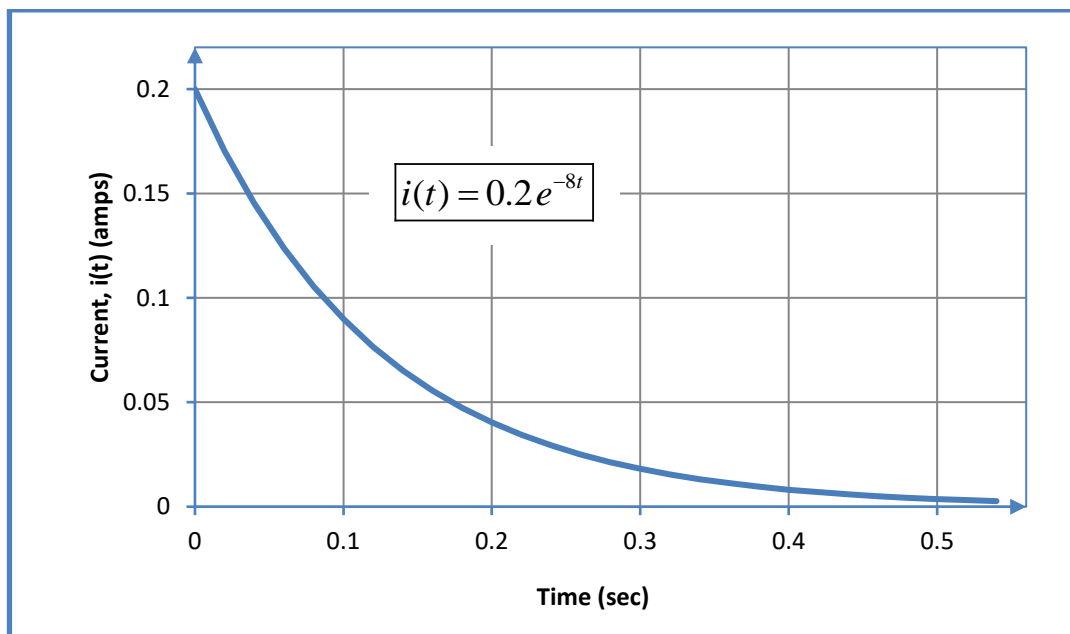
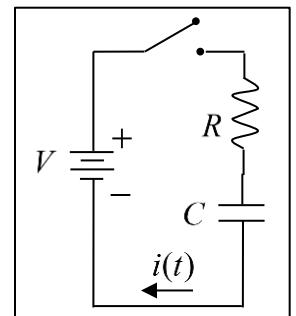
Given: For this circuit, $V = 100$ (volts), $R = 500(\Omega)$, and $C = 250(\mu\text{f})$.

When the switch is closed, it can be shown that the resulting

current is $i(t) = \left[\frac{V}{R} \right] e^{-t/RC}$.

Find: Calculate the exponential decay rate and plot $i(t)$.

Solution: The decay rate is $\alpha = -1/(500)(250 \times 10^{-6}) = -8$.



Example #3: Transient response of a direct current RL circuit

Given: For this circuit, $V = 100$ (volts), $R = 100(\Omega)$, and $L = 500(\text{mh})$.

When the switch is closed, it can be shown that the resulting

current is $i(t) = \left[\frac{V}{R} \right] \left(1 - e^{-t(R/L)} \right)$.

Find: Calculate the exponential decay rate and plot $i(t)$.

Solution: The decay rate is $\alpha = -100 / (500 \times 10^{-3}) = -200$.

