

Elementary Engineering Mathematics

Introduction to the Derivative of a Function

Consider a continuous function of a single variable $y = f(x)$. The **derivative** of the function at any point P is simply the **slope** of the function at that point. The line that has the same slope as $f(x)$ and passes through P is called the **tangent line** at that point.

We can find the slope of the tangent line using a **limiting process** as follows. First, define a **secant line** that passes through points $P(x_0, f(x_0))$ and $Q(x_0 + h, f(x_0 + h))$. Then, the slope of the secant line is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{secant line})$$

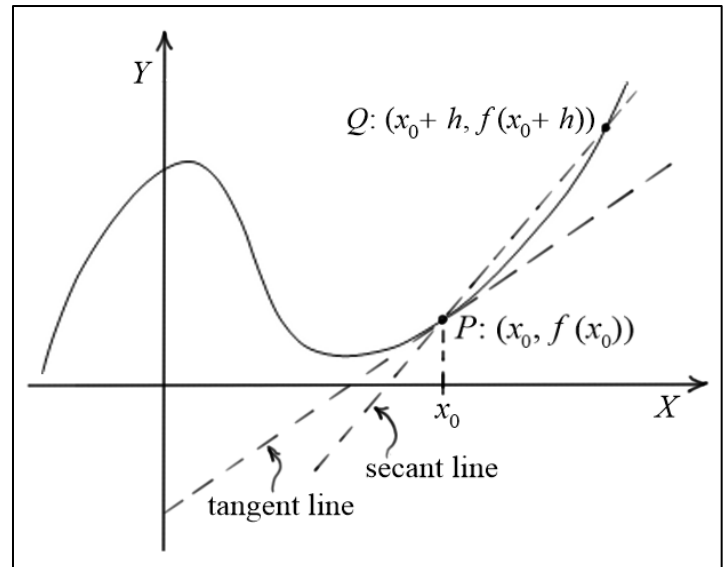
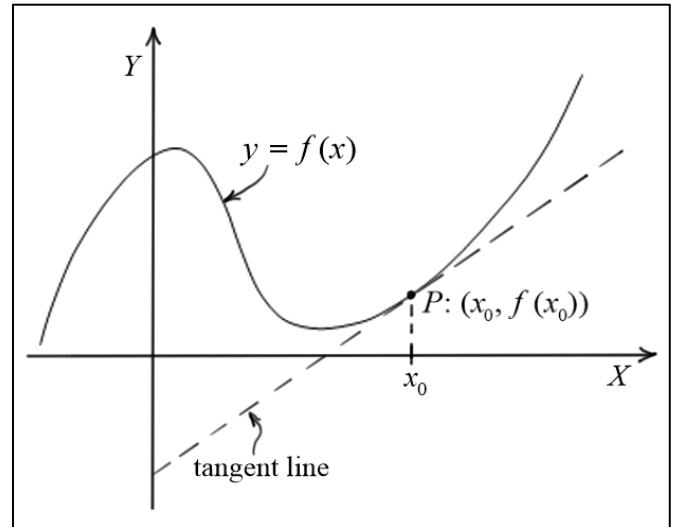
The slope of the tangent line is

$$\lim_{h \rightarrow 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} \right) \quad (\text{tangent line})$$

In the limit as Q moves to P , the slope of the secant line becomes the same as the slope of the tangent line.

If this limit exists, the function is said to be **differentiable** at x_0 , and the limit itself is called the **derivative** of f at x_0 . It is common to denote the derivative as

$$\lim_{h \rightarrow 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} \right) = \left. \frac{df}{dx} \right|_{x=x_0} = f'(x_0) \quad (1)$$



Once the slope is found, the *equation* of the *tangent line* can be found using the *point-slope form* for the equation of a line. The tangent line can be used as an *approximation* to the function f near x_0 .

Example 1:

Given: In previous notes, we found that the path of a golf ball (neglecting air resistance) was defined by the function

$$y = f(x) \approx 73.54 \left(\frac{x}{61.71} \right) - 16.1 \left(\frac{x}{61.71} \right)^2 \approx 1.1917x - (4.2278 \times 10^{-3})x^2 \quad (2)$$

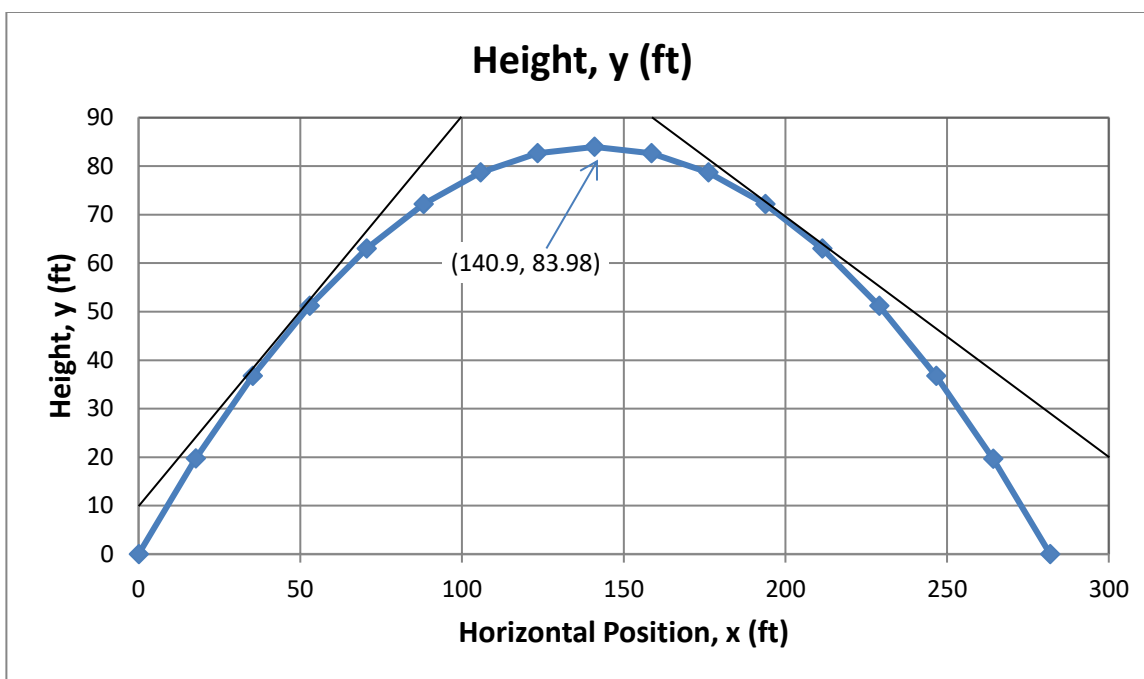


Figure 1. Height of Golf Ball as a Function of Distance, x

The *velocity* of the ball is in the direction of the tangent line.

Find: The *derivative* of the function $f(x)$ at a) $x_0 = 50$ (ft) and b) $x_0 = 200$ (ft). Use both graphical and analytical methods. Then find the *equation* of the *tangent line*, and the angle between the *velocity vector* and the X-axis for each case.

Solution:

a) Derivative at $x_0 = 50$:

Graphical method:

Using the plot in Fig. 1:
$$f'(x_0) = \frac{\Delta y}{\Delta x} \approx \frac{90 - 10}{100 - 0} = \frac{80}{100} = 0.8 \quad (3)$$

Analytical method:

$$f(x_0) = 1.1917(50) - (4.2278 \times 10^{-3})(50^2) = 59.585 - 10.5695 = 49.0155$$

$$\begin{aligned} f(x_0 + h) &= 1.1917(50 + h) - (4.2278 \times 10^{-3})((50 + h)^2) \\ &= 1.1917(50 + h) - (4.2278 \times 10^{-3})(50^2 + 100h + h^2) \\ &= [1.1917(50) - 4.2278 \times 10^{-3}(50^2)] + [1.1917(h) - 4.2278 \times 10^{-3}(100h + h^2)] \\ &= 49.0155 + (0.7689h - 4.2278 \times 10^{-3}(h^2)) \end{aligned}$$

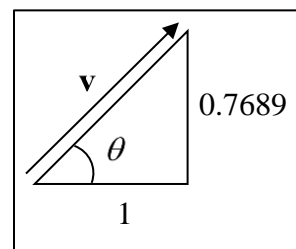
$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{(49.0155 + 0.7689h - (4.2278 \times 10^{-3})h^2) - 49.0155}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{0.7689h - (4.2278 \times 10^{-3})h^2}{h} \right) \\ &= \lim_{h \rightarrow 0} (0.7689 - (4.2278 \times 10^{-3})h) \\ &= 0.7689 \quad (\text{close to our graphical approximation}) \end{aligned} \quad (4)$$

We can find the equation of the tangent line at $x_0 = 50$ using the **point-slope form**:

$$\frac{y - y_0}{x - x_0} = m \quad \Rightarrow \quad \frac{y - 49.0155}{x - 50} = 0.7689 \quad \Rightarrow \quad y = 10.57 + 0.7689x \quad (5)$$

The angle between the velocity vector and the X-axis is found using the inverse tangent function.

$$\theta(x_0) = \tan^{-1}(\Delta y / \Delta x) = \tan^{-1}(0.7689) = \begin{cases} 37.56 \text{ (deg)} \\ 0.6555 \text{ (rad)} \end{cases}$$



b) Derivative at $x_0 = 200$:

Graphical method:

$$\text{Using the plot in Fig. 1: } f'(x_0) = \frac{\Delta y}{\Delta x} \approx \frac{20 - 90}{300 - 160} = \frac{-70}{140} = -0.5 \quad (6)$$

Analytical method:

$$f(x_0) = 1.1917(200) - (4.2278 \times 10^{-3})(200^2) = 238.34 - 169.112 = 69.228$$

$$\begin{aligned} f(x_0 + h) &= 1.1917(200 + h) - (4.2278 \times 10^{-3})((200 + h)^2) \\ &= 1.1917(200 + h) - (4.2278 \times 10^{-3})(200^2 + 400h + h^2) \\ &= [1.1917(200) - 4.2278 \times 10^{-3}(200^2)] \\ &\quad + [1.1917(h) - 4.2278 \times 10^{-3}(400h + h^2)] \\ &= 69.228 - (0.4994h + 4.2278 \times 10^{-3}(h^2)) \end{aligned}$$

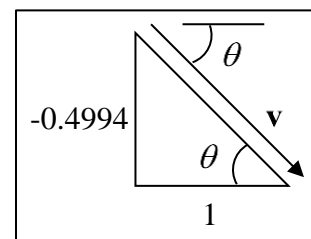
$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{(69.228 - 0.4994h - (4.2278 \times 10^{-3})h^2) - 69.228}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{-0.4994h - (4.2278 \times 10^{-3})h^2}{h} \right) \\ &= \lim_{h \rightarrow 0} (-0.4994 - (4.2278 \times 10^{-3})h) \\ &= -0.4994 \quad (\text{again close to our graphical approximation}) \end{aligned} \quad (7)$$

We can find the equation of the tangent line at $x_0 = 200$ by using the **point-slope form**:

$$\frac{y - y_0}{x - x_0} = m \Rightarrow \frac{y - 69.228}{x - 200} = -0.4994 \Rightarrow y = 169.11 - 0.4994x \quad (8)$$

The angle between the velocity vector and the X-axis is found using the inverse tangent function.

$$\theta(x_0) = \tan^{-1}(\Delta y / \Delta x) = \tan^{-1}(-0.4994) = \begin{cases} -26.54 \text{ (deg)} \\ -0.4632 \text{ (rad)} \end{cases}$$



Example 2:

Given: The tangent line equation at $x_0 = 50$ for the golf ball trajectory is given by Eq. (5) to be

$$y = 10.57 + 0.7689x$$

Find: Use this equation to generate approximate values for $f(x)$ in the range $30 \leq x \leq 70$.

Compare these values with those found using the exact equation given in Eq. (2). For each value, calculate the percent error of the approximation.

Solution:

The approximate values, actual values, and percent error are summarized in the table below.

x	y_{approx}	y_{actual}	% error = $100(y_{\text{approx}} - y_{\text{actual}})/y_{\text{actual}}$
30	33.6370	31.9460	5.29
35	37.4815	36.5304	2.60
40	41.3260	40.9035	1.03
45	45.1705	45.0652	0.23
50	49.0150	49.0155	≈ 0
55	52.8595	52.7544	0.20
60	56.7040	56.2819	0.75
65	60.5485	59.5980	1.59
70	64.3930	62.7028	2.70

