

Elementary Engineering Mathematics

The Derivative of a Function as a Function

Previously, we learned about the meaning of the derivative of a function $f(x)$ at some arbitrary point x_0 . The derivative $f'(x_0)$ is simply the slope of the tangent line at x_0 . We now consider the function $f'(x)$ or $\frac{df}{dx}(x)$ which consists of the derivatives of $f(x)$ at all points within the range of x . The following table gives the derivatives of some common functions used in engineering.

Name	Function, $f(x)$	Derivative, $f'(x) = \frac{df(x)}{dx}$
Constant	a	0
Polynomial terms	ax^n	nax^{n-1}
Exponential	e^{ax}	ae^{ax}
Sine	$\sin(ax)$	$a \cos(ax)$
Cosine	$\cos(ax)$	$-a \sin(ax)$

To evaluate the derivative at some point x_0 , we can simply evaluate the derivative function $f'(x)$ at x_0 .

These results can be extended to combinations of functions by using the following **rules** for differentiation.

	Name	Formula
1	Summation rule	$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
2	Multiplication by a constant, a	$\frac{d}{dx}(af(x)) = af'(x)$
3	Product rule	$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4	Chain rule	$\frac{d}{dx}(f(y(x))) = \frac{df}{dy} y'(x)$

Example 1:

Given: The path of a golf ball as a function of horizontal position

$$y = f(x) = 73.54 \left(\frac{x}{61.71} \right) - 16.1 \left(\frac{x}{61.71} \right)^2 = 1.1917x - (4.2278 \times 10^{-3})x^2 \quad (1)$$



Figure 1. Height of Golf Ball as a Function of Distance, x

Recall that the *velocity* of the ball is in the direction of the tangent line.

Find: (a) The *derivative function* $f'(x)$, (b) the derivative at $x = x_0 = 50$ (ft) and $x = x_0 = 200$ (ft), (c) the maximum height of the ball, (d) a plot of function $f'(x)$, and (e) the second derivative function $f''(x) = df'/dx$.

Solution:

(a) Using rule 1 above, we can find the derivative function $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(1.1917x - (4.2278 \times 10^{-3})x^2 \right) = \frac{d}{dx} (1.1917x) - \frac{d}{dx} \left((4.2278 \times 10^{-3})x^2 \right) \\ &= 1.1917 \frac{d}{dx} (x) - (4.2278 \times 10^{-3}) \frac{d}{dx} (x^2) \\ &= (1.1917 \times 1) - (4.2278 \times 10^{-3})(2x) \end{aligned}$$

or

$$f'(x) = 1.1917 - (8.4556 \times 10^{-3})x \quad (2)$$

(b) We can use the derivative function in Eq. (2) to find the derivative of the quadratic function at any point in the domain of x .

$$\text{At } x = x_0 = 50 \text{ (ft): } \boxed{f'(x)|_{x=50} = 1.1917 - (8.4556 \times 10^{-3})(50) \approx 0.76892 \approx 0.769}$$

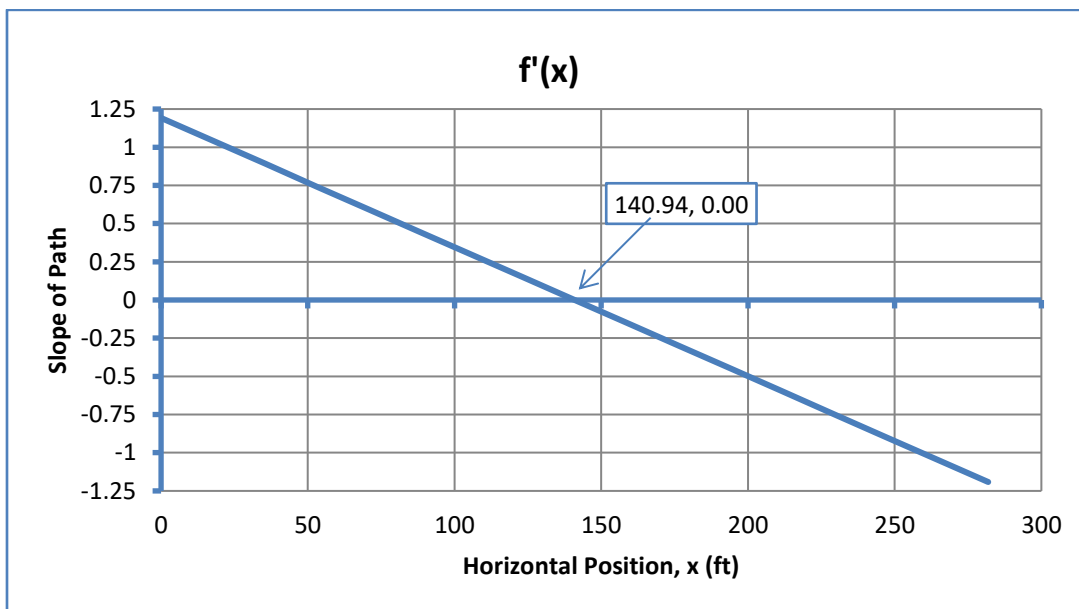
$$\text{At } x = x_0 = 200 \text{ (ft): } \boxed{f'(x)|_{x=200} = 1.1917 - (8.4556 \times 10^{-3})(200) \approx -0.49942 \approx -0.499}$$

(c) The maximum height of the ball occurs where the slope of the tangent line is zero.

$$f'(\hat{x}) = 0 = 1.1917 - (8.4556 \times 10^{-3})\hat{x} \Rightarrow \boxed{\hat{x} = 1.1917 / 8.4556e - 3 = 140.94 \text{ (ft)}}$$

$$y_{\max} = f(\hat{x}) = 1.1917 \hat{x} - (4.2278 \times 10^{-3})\hat{x}^2 \Rightarrow \boxed{y_{\max} = 83.977 \approx 84 \text{ (ft)}}$$

(d) The plot of $f'(x)$ indicates that the slope of $f(x)$ is positive over the first half of the range of x , negative over the second half, and zero at the maximum height of the ball. This confirms the fact that the function reached a **maximum** at this point, and **not a minimum**. When $f'(x)$ is positive, $f(x)$ is increasing, and when $f'(x)$ is negative, $f(x)$ is decreasing.

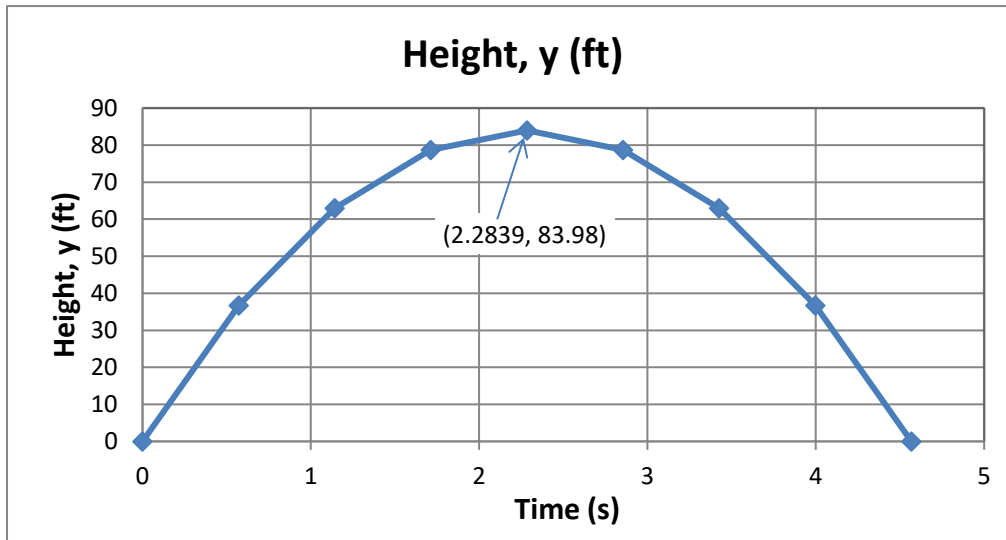


(e) Using rule 1, $\boxed{f''(x) = \frac{d}{dx} [1.1917 - (8.4556 \times 10^{-3})x] = -8.4556 \times 10^{-3} < 0}$. This again

confirms that the function is maximum at this point. If $f''(\hat{x}) < 0$, the function is at a **local maximum**, and if $f''(\hat{x}) > 0$, the function is at a **local minimum**.

Example 2:

Given: The height of a golf ball as a function of time $y = f(t) = 73.54t - 16.1t^2$



Find: (a) the derivative function $f'(t)$, (b) the slope of $f(t)$ at $t = 0.5$ (sec), and (c) the time \hat{t} when the ball reaches maximum height.

Solution:

(a) We can again use rule 1 to find $f'(t)$.

$$\frac{dy}{dt} = f'(t) = (73.54 \times 1) - 16.1(2t) = 73.54 - 32.2t$$

(b) When $t = 0.5$ (sec), $\left. \frac{dy}{dt} \right|_{t=0.5} = f'(t)|_{t=0.5} = 73.54 - (32.2 \times 0.5) \approx 57.44$ (ft/sec). This is the velocity of the ball in the Y-direction at this instant.

(c) To find time \hat{t} , we set $f'(\hat{t}) = 0$, and solve

$$f'(\hat{t}) = 0 = 73.54 - 32.2\hat{t} \Rightarrow \hat{t} = 73.54/32.2 \approx 2.2839 \approx 2.28 \text{ (sec)}$$

Example 3:

Given: The horizontal position and height of a golf ball are functions of time

$$x(t) = 61.71t$$

$$y(t) = 73.54t - 16.1t^2$$

(3)

Find: The velocity vector of the ball when $x = x_0 = 50$ (ft).

Solution:

The components of the velocity of the ball in the X and Y directions are given by the derivatives of Eqs. (3) with respect to time. We must first find the time \hat{t} required to get to $x = x_0 = 50$ (ft).

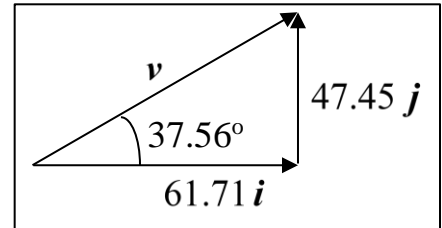
$$x(\hat{t}) = 50 = 61.71\hat{t} \Rightarrow \boxed{\hat{t} = 50 / 61.71 \approx 0.81024 \text{ (sec)}}$$

$$\boxed{v_x(t) = \frac{dx}{dt} = \dot{x}(t) = 61.71 \times (1) \approx 61.71 \text{ (ft/sec)}} \quad (\text{same at all } t)$$

$$\boxed{v_y(t) = \frac{dy}{dt} = \dot{y}(t) = 73.54 - 32.2t}$$

$$\Rightarrow \boxed{v_y(\hat{t}) = 73.54 - (32.2 \times 0.81024) \approx 47.45 \text{ (ft/sec)}}$$

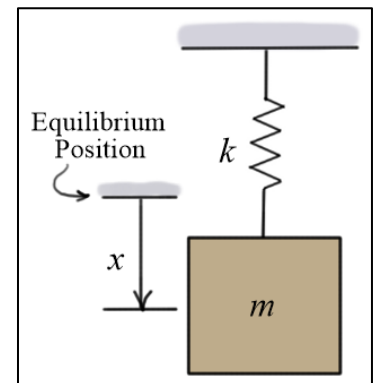
So, at $x = x_0 = 50$ (ft), we have $\boxed{\underline{v} = 61.71\hat{i} + 47.45\hat{j} \text{ (ft/sec)}}$



Example 4: Undamped, free vibration

Given: In response to the initial position x_0 and initial velocity v_0 , the undamped spring-mass-damper system has displacement function

$$\boxed{x(t) = \frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t)} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$



Find: (a) the velocity function $v(t) = \dot{x}(t)$, (b) the acceleration function $a(t) = \dot{v}(t) = \ddot{x}(t)$, (c) $x(0)$, $v(0)$, and $a(0)$, the position, velocity and acceleration of the mass at $t = 0$, and (d) the times when $x(t)$ has a maximum or minimum if $v_0 = 0$, and verify which are maxima and which are minima.

Solution:

(a) Using rules 1 and 2:

$$\begin{aligned} v(t) = \dot{x}(t) &= \frac{d}{dt} \left[\frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t) \right] = \frac{d}{dt} \left[\frac{v_0}{\omega} \sin(\omega t) \right] + \frac{d}{dt} \left[x_0 \cos(\omega t) \right] \\ &= \frac{v_0}{\cancel{\omega}} \cancel{\omega} \cos(\omega t) - x_0 \omega \sin(\omega t) \end{aligned}$$

$$\boxed{v(t) = v_0 \cos(\omega t) - x_0 \omega \sin(\omega t)}$$

(b) Again, using rules 1 and 2:

$$a(t) = \ddot{x}(t) = \dot{v}(t) = \frac{d}{dt} [v_0 \cos(\omega t) - x_0 \omega \sin(\omega t)]$$

$$= \frac{d}{dt} [v_0 \cos(\omega t)] - \frac{d}{dt} [x_0 \omega \sin(\omega t)]$$

$$a(t) = -v_0 \omega \sin(\omega t) - x_0 \omega^2 \cos(\omega t)$$

(c) $x(0) = \frac{v_0}{\omega} \sin(0) + x_0 \cos(0) = x_0$ $v(0) = v_0 \cos(0) - x_0 \omega \sin(0) = v_0$ (checks)

$$a(0) = -v_0 \omega \sin(0) - x_0 \omega^2 \cos(0) = -x_0 \omega^2$$

(d) If $v_0 = 0$, the velocity and acceleration of the mass are $v(t) = -x_0 \omega \sin(\omega t)$ and

$a(t) = -x_0 \omega^2 \cos(\omega t)$. The position has a maximum or minimum at times \hat{t} when the velocity

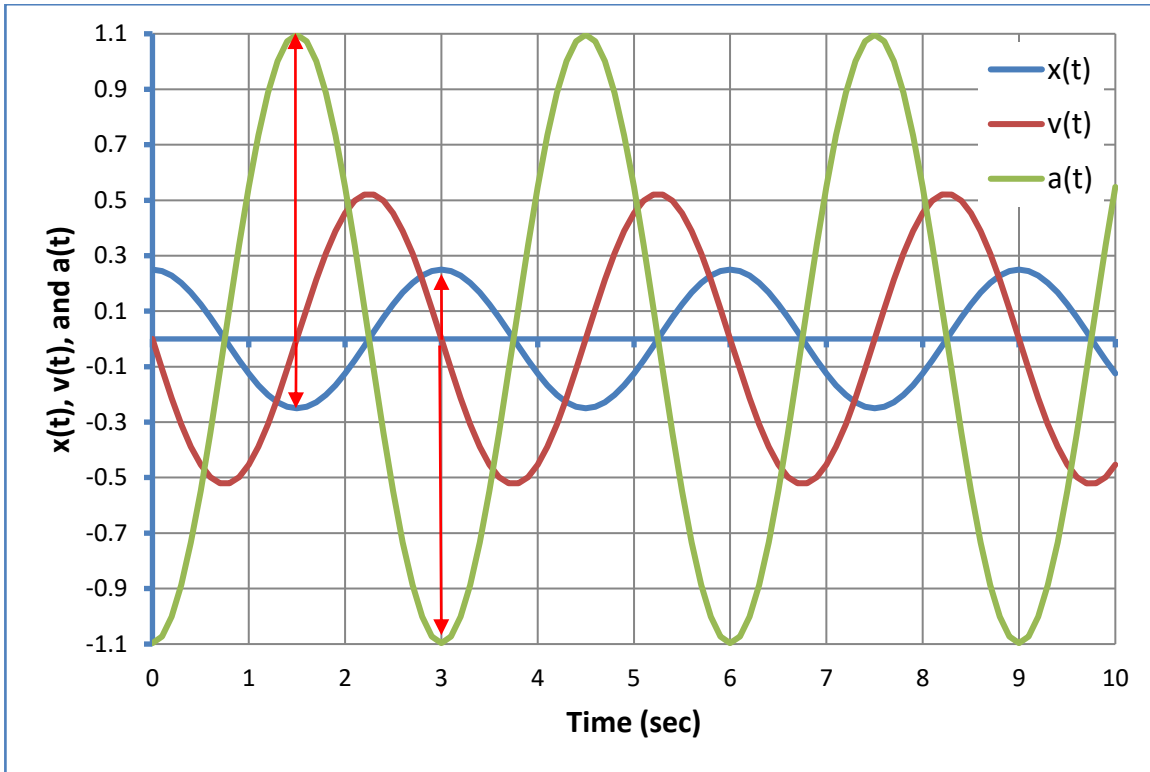
$v(\hat{t}) = 0$. So, the position will be a maximum or minimum when $\omega \hat{t} = n\pi$ or $\hat{t} = n\pi/\omega$

($n = 0, 1, 2, \dots$). The results are summarized in the following table.

n	$v(\hat{t})$	$a(\hat{t})$	Type
0, 2, 4, ...	zero	negative	maximum
1, 3, 5, ...	zero	positive	minimum

The figure below shows the position, velocity, and acceleration functions for $x_0 = 0.25$ (ft) and $\omega = 2\pi/3$ (rad/s). When the velocity is zero, the displacement is either a maximum or minimum.

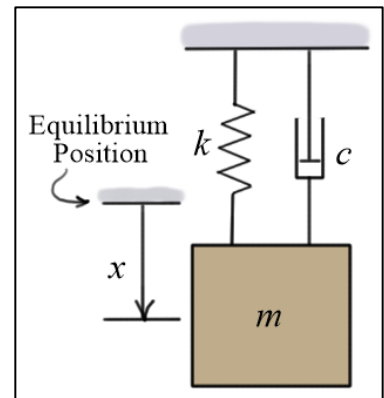
It is a maximum when the acceleration is negative, and it is a minimum when the acceleration is positive (as indicated in the table above).



Example 5: Over-damped, free vibration

Given: In response to the initial conditions $x_0 = 0.25$ (ft) and $v_0 = 5$ (ft/s), the over-damped spring-mass-damper system has displacement

$$x(t) = \left((0.5163)e^{-3.82t} - (0.2663)e^{-26.18t} \right) \text{ (ft)}$$



Find: (a) the velocity function $v(t) = \dot{x}(t)$; (b) the acceleration function $a(t) = \dot{v}(t) = \ddot{x}(t)$; (c) $x(0)$, $v(0)$, and $a(0)$, the position, velocity and acceleration of the mass at $t = 0$; and (d) find the time when the displacement is maximum.

Solution:

(a) Using rules 1 and 2, we find $v(t)$

$$\begin{aligned} v(t) &= \dot{x}(t) = \frac{d}{dt} \left((0.5163)e^{-3.82t} - (0.2663)e^{-26.18t} \right) \\ &= \frac{d}{dt} \left[(0.5163)e^{-3.82t} \right] - \frac{d}{dt} \left[(0.2663)e^{-26.18t} \right] \\ &= (0.5163)(-3.82)e^{-3.82t} - (0.2663)(-26.18)e^{-26.18t} \end{aligned}$$

$$v(t) = -1.9723e^{-3.82t} + 6.9717e^{-26.18t} \text{ (ft/s)}$$

(b) Using rules 1 and 2 again, we find $a(t)$

$$a(t) = \ddot{x}(t) = \dot{v}(t) = \frac{d}{dt} \left[-1.9723e^{-3.82t} + 6.9717e^{-26.18t} \right]$$

$$= (-1.9723)(-3.82)e^{-3.82t} + (6.9717)(-26.18)e^{-26.18t}$$

$$a(t) = 7.5342e^{-3.82t} - 182.52e^{-26.18t} \text{ (ft/s}^2\text{)}$$

(c) $x(0) = \left((0.5163)e^0 - (0.2663)e^0 \right) = 0.5163 - 0.2663 = 0.25 \text{ (ft)}$ (checks)

$$v(0) = -1.9723e^0 + 6.9717e^0 = 6.9717 - 1.9723 = 4.9994 \approx 5 \text{ (ft/s)}$$
 (checks)

$$a(0) = 7.5342e^0 - 182.52e^0 = 7.5342 - 182.52 \approx -175 \text{ (ft/s}^2\text{)}$$

(d) To find the time when the displacement is maximum, we set $dx(t)/dt = v(t) = 0$.

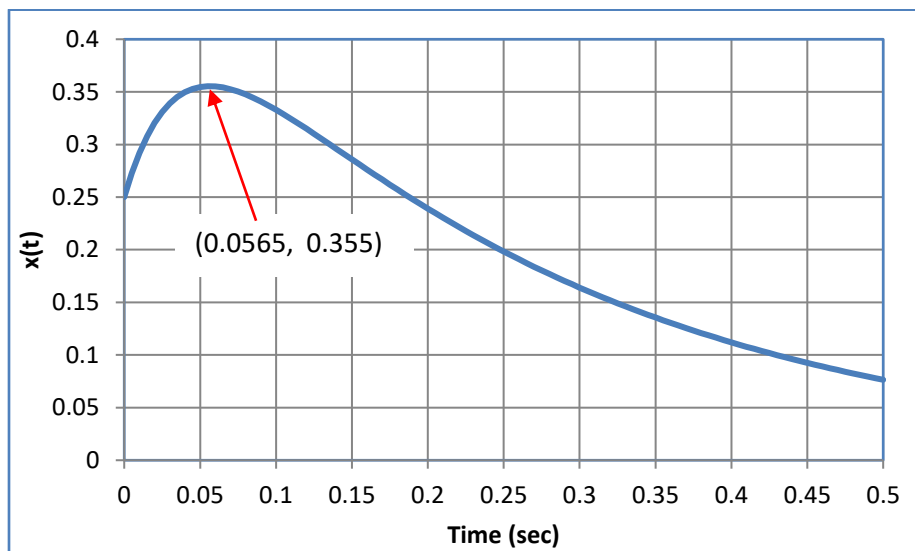
$$v(t) = -1.9723e^{-3.82t} + 6.9717e^{-26.18t} = 0 \quad \Rightarrow \quad 6.9717e^{-26.18t} = 1.9723e^{-3.82t}$$

$$\Rightarrow \frac{6.9717}{1.9723} = 3.5348 = \frac{e^{-3.82t}}{e^{-26.18t}} = e^{22.36t} \quad \Rightarrow \quad \ln(3.5348) = \ln(e^{22.36t}) = 22.36t$$

$$t = \ln(3.5348)/22.36 = 0.0565 \text{ (sec)}$$

$$x(0.0565) = (0.5163)e^{-0.21583} - (0.2663)e^{-1.4792} = 0.355 \text{ (ft)}$$

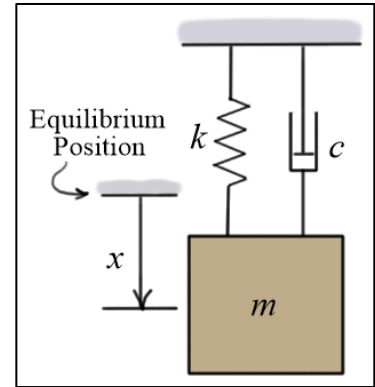
$$a(0.0565) = 7.5342e^{-0.21583} - 182.52e^{-1.4792} = -35.5 \text{ (ft/s}^2\text{)}$$
 (indicates maximum)



Example 6: Under-damped, free vibration

Given: In response to the initial conditions $x_0 = 0.25$ (ft) and $v_0 = 5$ (ft/s), the under-damped spring-mass-damper system has displacement

$$x(t) = e^{-5t} \left[0.7217 \sin(\sqrt{75} t) + 0.25 \cos(\sqrt{75} t) \right] \text{ (ft)}$$



Find: (a) the velocity function $v(t) = \dot{x}(t)$, (b) the acceleration function $a(t) = \dot{v}(t) = \ddot{x}(t)$, and (c) $x(0)$, $v(0)$, and $a(0)$, the position, velocity and acceleration of the mass at $t = 0$.

Solution:

(a) Using rules 2, 3 and 4, we find $v(t)$

$$\begin{aligned} v(t) = \dot{x}(t) &= \frac{d}{dt} \left(e^{-5t} \left[0.7217 \sin(\sqrt{75} t) + 0.25 \cos(\sqrt{75} t) \right] \right) \\ &= \frac{d}{dt} (e^{-5t}) \times \left[0.7217 \sin(\sqrt{75} t) + 0.25 \cos(\sqrt{75} t) \right] \\ &\quad + e^{-5t} \left(\frac{d}{dt} \left[0.7217 \sin(\sqrt{75} t) + 0.25 \cos(\sqrt{75} t) \right] \right) \\ &= -5e^{-5t} \left[0.7217 \sin(\sqrt{75} t) + 0.25 \cos(\sqrt{75} t) \right] \\ &\quad + e^{-5t} \left((0.7217)\sqrt{75} \cos(\sqrt{75} t) - (0.25)\sqrt{75} \sin(\sqrt{75} t) \right) \\ &= e^{-5t} \left[-\left((5 \times 0.7217) + (0.25 \times \sqrt{75}) \right) \sin(\sqrt{75} t) \right] \\ &\quad + e^{-5t} \left[\left((0.7217\sqrt{75}) - (5 \times 0.25) \right) \cos(\sqrt{75} t) \right] \end{aligned}$$

$$v(t) = e^{-5t} \left[-5.7736 \sin(\sqrt{75} t) + 5 \cos(\sqrt{75} t) \right] \text{ (ft/s)}$$

(b) Using rules 2, 3 and 4, we find $a(t)$

$$\begin{aligned} a(t) &= \ddot{x}(t) = \dot{v}(t) = \frac{d}{dt} \left(e^{-5t} \left[-5.7736 \sin(\sqrt{75} t) + 5 \cos(\sqrt{75} t) \right] \right) \\ &= \frac{d}{dt} (e^{-5t}) \times \left[-5.7736 \sin(\sqrt{75} t) + 5 \cos(\sqrt{75} t) \right] \\ &\quad + e^{-5t} \left(\frac{d}{dt} \left[-5.7736 \sin(\sqrt{75} t) + 5 \cos(\sqrt{75} t) \right] \right) \\ &= -5e^{-5t} \left[-5.7736 \sin(\sqrt{75} t) + 5 \cos(\sqrt{75} t) \right] \\ &\quad + e^{-5t} \left((-5.7736\sqrt{75}) \cos(\sqrt{75} t) - (5)\sqrt{75} \sin(\sqrt{75} t) \right) \\ &= e^{-5t} \left[(5 \times 5.7736) - (5\sqrt{75}) \right] \sin(\sqrt{75} t) \\ &\quad + e^{-5t} \left[-\left((5.7736\sqrt{75}) + (5 \times 5) \right) \cos(\sqrt{75} t) \right] \end{aligned}$$

$$a(t) = -e^{-5t} \left[14.43 \sin(\sqrt{75} t) + 75 \cos(\sqrt{75} t) \right] \text{ (ft/s}^2\text{)}$$

(c) $x(0) = e^0 \left[0.7217 \sin(0) + 0.25 \cos(0) \right] = 0.25 \text{ (ft)}$ (checks)

$$v(0) = e^0 \left[-5.7736 \sin(0) + 5 \cos(0) \right] = 5 \text{ (ft/s)}$$
 (checks)

$$a(0) = -e^0 \left[14.43 \sin(0) + 75 \cos(0) \right] = -75 \text{ (ft/s}^2\text{)}$$