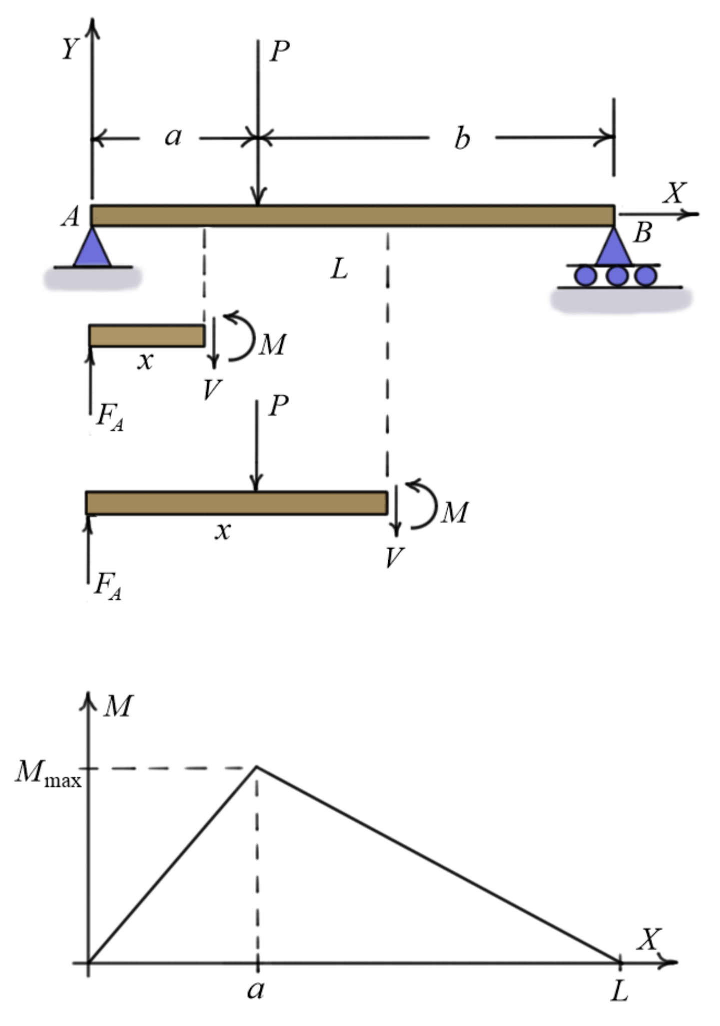
**Elementary Engineering Mathematics**
**Applications of Derivatives in Statics, Mechanics of Materials**

Example #1

Consider a ***long slender beam*** of length $L$ with a ***concentrated load*** $P$ acting at distance $a$ from the left end. Due to this load, the beam experiences an ***internal bending moment*** $M(x)$ and ***internal shearing force*** $V(x)$. As presented in earlier notes, the bending moment is zero at both ends of the beam and rises linearly from there to a maximum value at $x = a$. The shearing force is the derivative of the bending moment.




$$\boxed{V(x) = \frac{dM(x)}{dx} = M'(x)}$$

Given: $P = 100$ (lbs), $L = 5$ (ft), $a = 3.5$ (ft), and $M_{\text{max}} = abP/L$

Find: (a) $M(x)$ for $0 \le x \le L$; (b) $V(x)$ for $0 \le x \le L$; and (c) plot the functions.

Solution: $M_{\text{max}} = abP/L = (3.5)(1.5)100/5 = 105$ (ft-lb)

(a) For $(0 \le x \le a)$, the slope is $\boxed{m = (105-0)/(3.5-0) = 30 \text{ (ft-lb/ft)}}$.

$$\boxed{M(x) = 30x \text{ (ft-lb)}}$$

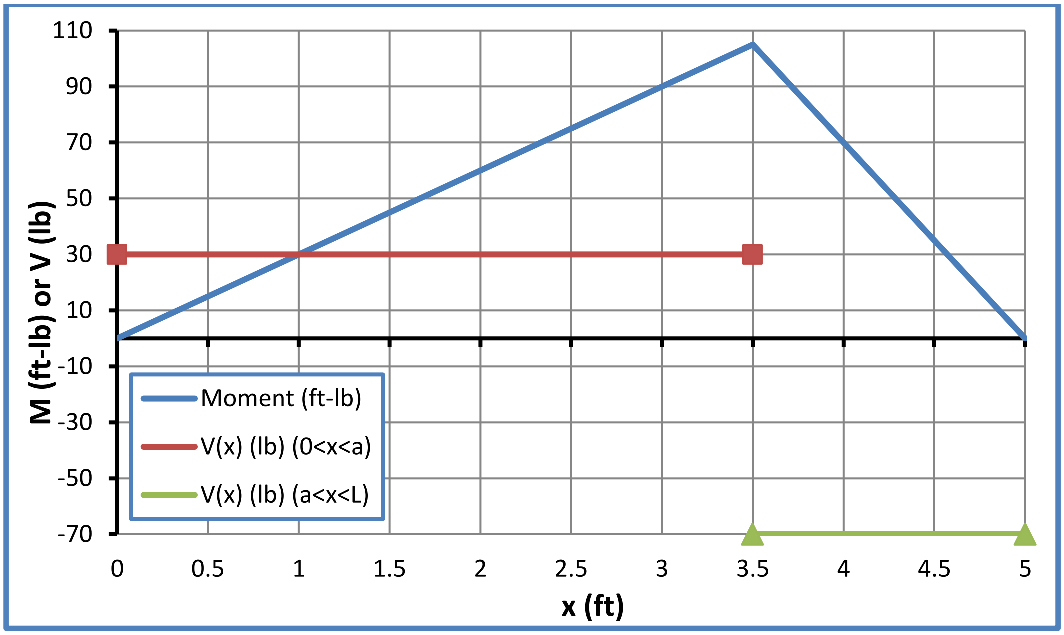
For $(a \le x \le L)$, the slope is $\boxed{m = (0-105)/(5-3.5) = -70 \text{ (ft-lb/ft)}}$. Using the point-slope form

$$(M - 105) = -70(x - 3.5) \quad \Rightarrow \quad \boxed{M(x) = 350 - 70x \text{ (ft-lb)}}$$

(b) For $(0 \le x \le a)$, $\boxed{V(x) = M'(x) = \frac{d}{dx}(30x) = 30 \text{ (lb)}}$

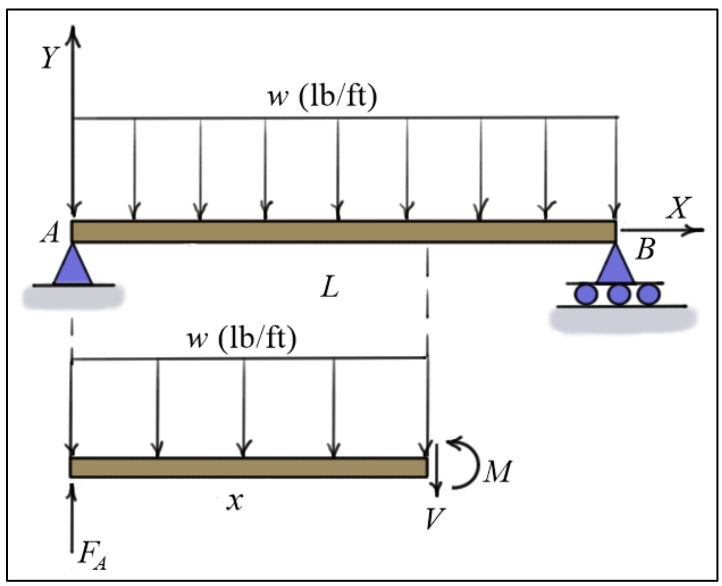
For $(a \le x \le L)$, $\boxed{V(x) = M'(x) = \frac{d}{dx}(350 - 70x) = \frac{d}{dx}(350) + \frac{d}{dx}(-70x) = -70 \text{ (lb)}}$

(c) Plot of the shear force and bending moment along the beam.




Question: What is the value of $M'(x)$ at $x = 3.5$ (ft)?

Example 2:




Given: $L = 10$ (ft), $w = 100$ (lb/ft), and

$$\boxed{M(x) = 500x - 50x^2 \text{ (ft-lb)}} \quad (0 \le x \le L)$$

Find: (a) shearing force $V(x)$; (b) maximum bending moment and its location; and (c) plot $M(x)$ and $V(x)$.

Solution:

(a) For $(0 \le x \le L)$

$$\boxed{V(x) = M'(x) = \frac{d}{dx}\left(500x - 50x^2\right) = \frac{d}{dx}(500x) + \frac{d}{dx}\left(-50x^2\right) = 500 - 100x \text{ (lb)}}$$

(b) Because the shearing force is continuous, the bending moment is a maximum (or minimum) either at an end of the beam or where the shear ***zero***.

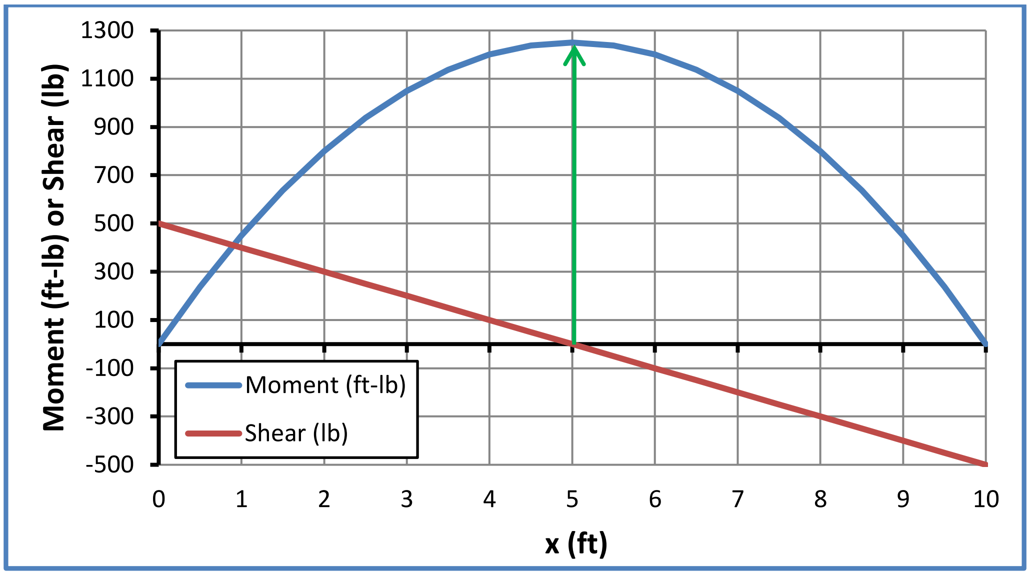
$$V(x) = M'(x) = 500 - 100x = 0 \quad \Rightarrow \boxed{x = 500/100 = 5 \text{ (ft)}}$$

$$\boxed{M(0)=M(L)=0} \text{ and } \boxed{M(x=5)=(500\times 5)-(50\times 5^2)=1250 \text{ (ft-lb)}=M_{max}}$$

To verify that it is a maximum, check the sign of $M''(x)$:

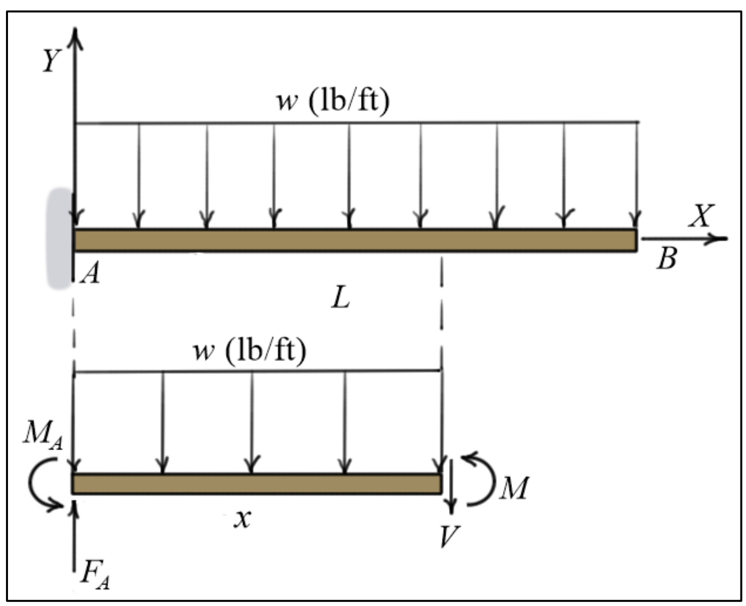
$$\boxed{M''(x)=\frac{d}{dx}(500-100x)=-100<0} \text{ (it is a maximum)}$$

(c) Plot of the shear force and bending moment along the beam.




Example 3:

Consider a cantilevered beam with a ***uniformly distributed load*** of $w$ (lb/ft). If the beam is cut at a distance $x$ from the wall, we expose the internal ***shearing force*** $V$ and ***bending moment*** $M$.




Given: $L=10$ (ft), $w=100$ (lb/ft), and

$$\boxed{M(x)=-\tfrac{1}{2}wx^2+wLx-\tfrac{1}{2}wL^2 \text{ (ft-lb)}}$$

Find: (a) the shearing force $V(x)$; (b) the maximum bending moment and its location; and (c) plot $M(x)$ and $V(x)$.

Solution: Using the values for $L$ and $w$, $\boxed{M(x) = -50x^2 + 1000x - 5000 \text{ (ft-lb)}}$

(a) $\boxed{V(x) = M'(x) = \frac{d}{dx}\left(-50x^2 + 1000x - 5000\right) = 1000 - 100x \text{ (lb)}}$

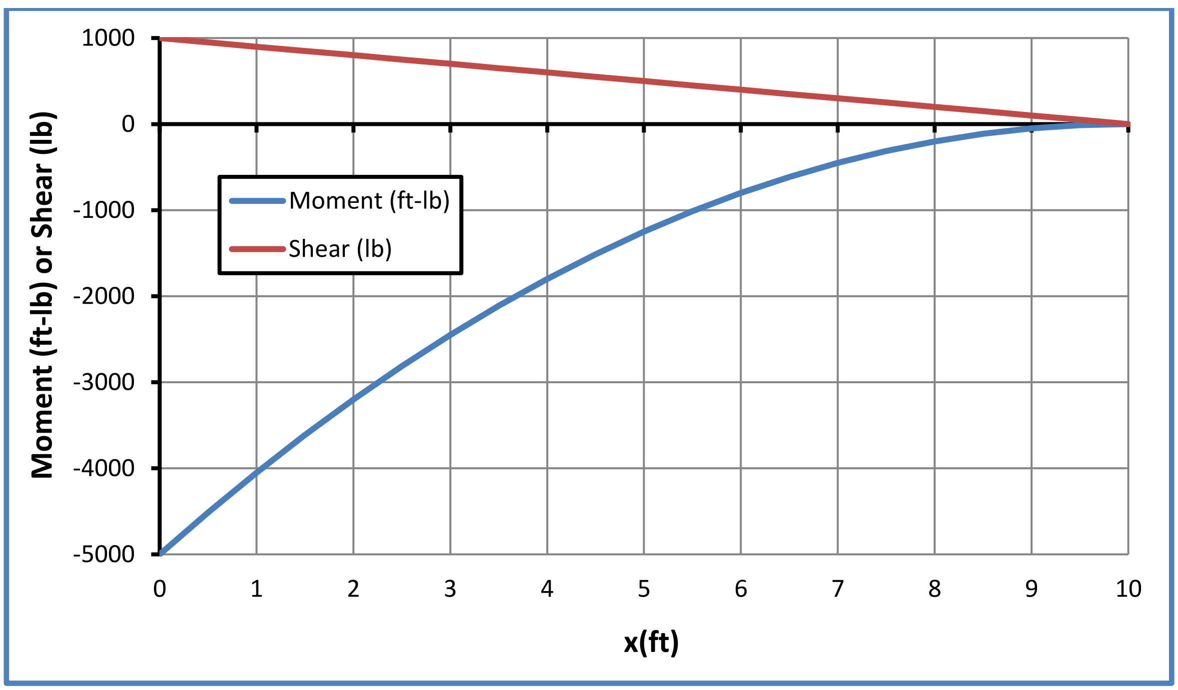
(b) Again, the shearing force is continuous, so the bending moment is a maximum (or minimum) either at an end of the beam or where the shear ***zero***.

$V(x) = M'(x) = 1000 - 100x = 0 \quad \Rightarrow \boxed{x = 1000/100 = 10 \text{ (ft)}}$ (at the end)

$\boxed{M(0) = -5000 \text{ (ft-lb)}}$ $\boxed{M(10) = 0 \text{ (ft-lb)}}$ $\Rightarrow \boxed{M_{\max} = -5000 \text{ (ft-lb)}}$

In this case, the maximum occurs at the end of the beam, and not where $M'(x) = 0$, because our concern is with the absolute value of the bending moment. We must design the beam to withstand 5000 (ft-lb) of bending moment, not zero.

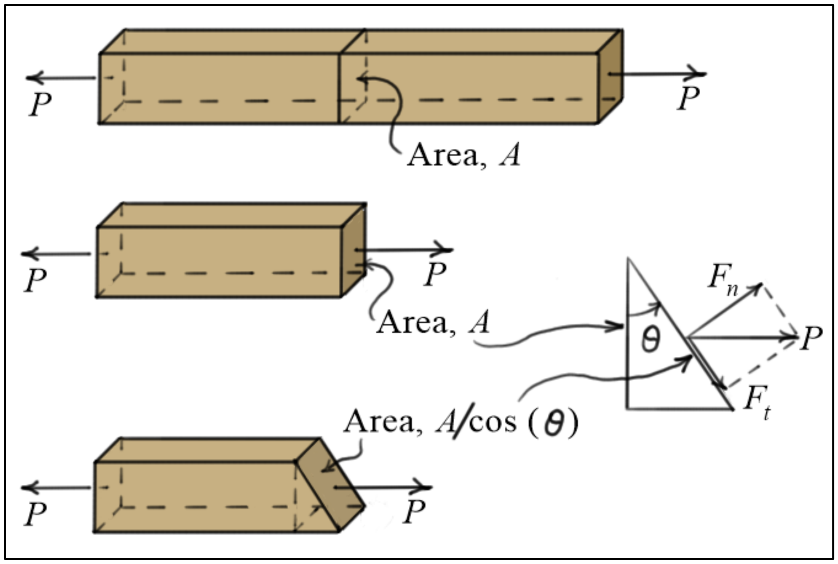
(c) Plot of the shear force and bending moment along the beam.



Example 4:

Consider a bar with rectangular cross-sectional area $A$ and applied force $P$ as shown. Because $A$ is perpendicular (or normal) to the direction of $P$, the material on $A$ experiences ***normal stress*** only and is defined as

$$\boxed{\sigma = P/A}$$




Now consider a plane at an angle $\theta$ to the vertical. Since this plane is not normal to $P$, the material along this plane experiences both ***normal stress*** and ***shear stress***.

The normal stress $\sigma$ is defined as the ratio of the normal force and the area. The shear stress $\tau$ is defined as the ratio of the tangential force and the area.

$$\boxed{\sigma = \frac{F_n}{A/\cos(\theta)} = \frac{P\cos(\theta)}{A/\cos(\theta)} = \left(\frac{P}{A}\right)\cos^2(\theta)} \qquad \boxed{\tau = \frac{F_t}{A/\cos(\theta)} = \frac{P\sin(\theta)}{A/\cos(\theta)} = \left(\frac{P}{A}\right)\sin(\theta)\cos(\theta)}$$

In a simple tension test, such as that described above, ***brittle materials*** tend to fail due to excessive ***normal stress***, and ***ductile materials*** tend to fail due to excessive ***shear stress***.

By thinking of the normal and shear stresses as functions of the cut angle angle $\theta$, we can find which planes experience the ***highest*** normal and shear stresses. We can find maxima and minima by setting $d\sigma/d\theta = 0$ and $d\tau/d\theta = 0$ and solving for the angle $\theta$. Using the product and chain rules gives

$$\boxed{d\sigma/d\theta = \frac{d}{d\theta}\left[(P/A)\cos^2(\theta)\right] = (P/A)(2\cos(\theta))(-\sin(\theta)) = -(2P/A)\sin(\theta)\cos(\theta)}$$

$$\boxed{d^2\sigma/d\theta^2 = \frac{d}{d\theta}\left[-(2P/A)\sin(\theta)\cos(\theta)\right] = (2P/A)\left(\sin^2(\theta) - \cos^2(\theta)\right)}$$

$$\boxed{d\tau/d\theta = \frac{d}{d\theta}\left[(P/A)\sin(\theta)\cos(\theta)\right] = (P/A)\left[\cos^2(\theta) - \sin^2(\theta)\right] = (P/A)\cos(2\theta)}$$

$$d^2\tau/d\theta^2 = \frac{d}{d\theta}\Big[(P/A)\cos(2\theta)\Big] = (P/A)\Big[(-\sin(2\theta))(2)\Big]$$
$$= (-2P/A)\sin(2\theta)$$

Setting the derivatives to zero and considering $\boxed{0 \le \theta < \pi/2}$, we get the following results.

| Stress | Angle, $\theta$ | 1st Derivative | 2nd Derivative | Type |
|---|---|---|---|---|
| $\sigma$ | 0 | 0 | negative | maximum |
| $\tau$ | $\pi/4$ (rad) $= 45^{\circ}$ | 0 | negative | maximum |

So, ***brittle materials*** will be more likely to fail on a plane ***normal to*** the load, and ***ductile materials*** will be more likely to fail on a ***45º plane***.