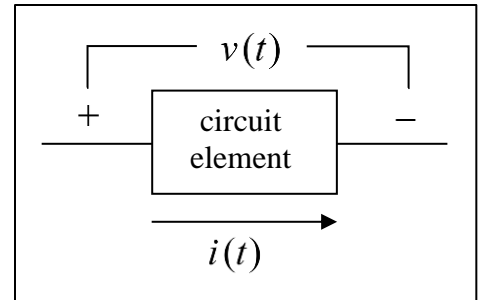


Elementary Engineering Mathematics

Application of Derivatives in Electrical Engineering

The diagram shows a typical element (resistor, capacitor, inductor, etc.) in an electrical circuit. Here, $v(t)$ represents the voltage across the element, and $i(t)$ represents the current flowing through the element. Both generally are functions of time, t . For any such element, the following equations apply.



$$\boxed{v(t) = \frac{dw}{dq}} \quad \left\{ \begin{array}{l} v(t) \text{ is the voltage (volts)} \\ w(t) \text{ is the energy (joules)} \\ q(t) \text{ is the charge (coulombs)} \end{array} \right.$$

$$\boxed{i(t) = \frac{dq}{dt}} \quad \left\{ \begin{array}{l} i(t) \text{ is the current (amps)} \\ t \text{ is the time (sec)} \end{array} \right.$$

$$\boxed{p(t) = \frac{dw}{dt} = \left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right) = v(t) \cdot i(t)} \quad \{ p(t) \text{ is the power (joules/sec) or (watts)} \}$$

This last equation is an application of the *chain rule*.

Example 1:

Given: The charge and voltage for a given circuit element are given by the following equations:

$$\boxed{q(t) = \frac{1}{50} \sin(250\pi t) \text{ (coulombs)}} \quad \text{and} \quad \boxed{v(t) = 100\sin(250\pi t) \text{ (volts)}}$$

Find: a) $i(t)$ the current passing through the element, b) $p(t)$ the power dissipated by the element, and c) p_{\max} the maximum power dissipated by the element.

Solution:

$$\text{a) } i(t) = \frac{dq}{dt} = \frac{d}{dt} \left(\frac{1}{50} \sin(250\pi t) \right) = \frac{1}{50} \times \cos(250\pi t) \times 250\pi = \boxed{5\pi \cos(250\pi t) \text{ (amps)}}$$

$$\begin{aligned} \text{b) } p(t) &= v(t) \cdot i(t) = (100\sin(250\pi t))(5\pi \cos(250\pi t)) \\ &= \boxed{500\pi \sin(250\pi t) \cos(250\pi t) \text{ (watts)}} \end{aligned}$$

$$\text{c) To find maximum power, we use the trigonometric identity: } \boxed{2\sin(\theta)\cos(\theta) = \sin(2\theta)}$$

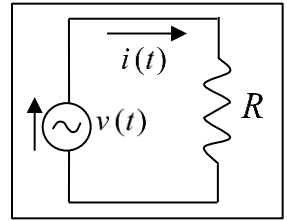
$$p(t) = 500\pi \sin(250\pi t) \cos(250\pi t) = \frac{1}{2} (500\pi \sin(500\pi t)) = \boxed{250\pi \sin(500\pi t) \text{ (watts)}}$$

$$\Rightarrow \boxed{p_{\max} = 250\pi \approx 785 \text{ (watts)}}$$

Current–Voltage Relationships for Resistors, Capacitors, and Inductors

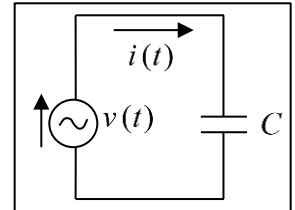
The voltage across and the current through a *resistor* are related simply by its resistance.

$$\boxed{v(t) = Ri(t)} \quad \text{or} \quad \boxed{i(t) = v(t)/R}$$



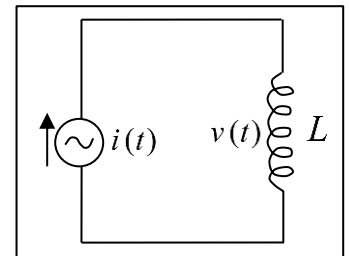
Given a voltage $v(t)$ applied to a *capacitor*, the corresponding current $i(t)$ can be calculated as

$$\boxed{i(t) = C \frac{dv}{dt}}$$



Given the current $i(t)$ passing through an *inductor*, the corresponding voltage $v(t)$ can be calculated as

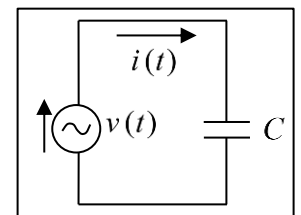
$$\boxed{v(t) = L \frac{di}{dt}}$$



Example 2:

Given: A voltage $v(t) = 110\cos(120\pi t)$ (volts) is applied to a capacitor with $C = 100$ (μf).

Find: a) $i(t)$, the current through the capacitor, and b) the maximum power p_{\max} .



Solution:

a) The current may be found by differentiating the voltage.

$$\begin{aligned} i(t) &= C \frac{dv}{dt} = (100 \times 10^{-6}) \frac{d}{dt} (110\cos(120\pi t)) \\ &= (100 \times 10^{-6})(110)(120\pi)(-\sin(120\pi t)) \\ &= \boxed{-4.147 \sin(120\pi t) \text{ (amps)}} \end{aligned}$$

$$\begin{aligned} \text{b) } p(t) &= v(t) \cdot i(t) = (110\cos(120\pi t))(-4.147 \sin(120\pi t)) \\ &= -456 \sin(120\pi t) \cos(120\pi t) \\ &= -456 \times \frac{1}{2} \sin(240\pi t) \\ &= \boxed{-228 \sin(240\pi t) \text{ (watts)}} \Rightarrow \boxed{p_{\max} = 228 \text{ (watts)}} \end{aligned}$$

Alternate Solution for part (a): (without using derivatives)

We could have solved this problem using *complex numbers*.

$$Z_C = -j/\omega C = -j/(120\pi)(100 \times 10^{-6}) = -j(10^6)/(120\pi)(100) \\ \approx -j26.526 \text{ (ohms)} \approx 26.526 \angle (-90^\circ)$$

$$I \approx \frac{110 \angle (0^\circ)}{26.526 \angle (-90^\circ)} \approx 4.147 \angle (90^\circ)$$

$$\Rightarrow i(t) \approx 4.147 \cos(120\pi t + 90^\circ) \text{ (amps)}$$

$$\approx 4.147 \left(\cos(120\pi t) \underbrace{\cos(90^\circ)}_0 - \sin(120\pi t) \underbrace{\sin(90^\circ)}_1 \right)$$

$$\Rightarrow i(t) \approx -4.147 \sin(120\pi t) \text{ (amps)}$$

Example 3:

Given: A voltage $v(t) = 110e^{-10t} \cos(120\pi t)$ (volts) is applied to a capacitor with $C = 100$ (μf).

Find: $i(t)$, the current through the capacitor

Solution:

To find the current we differentiate the voltage using the product and chain rules.

$$i(t) = C \frac{dv}{dt} = (100 \times 10^{-6}) \frac{d}{dt} (110e^{-10t} \cos(120\pi t)) \\ = (100 \times 10^{-6})(110) \left[\left(\frac{d}{dt} (e^{-10t}) \times (\cos(120\pi t)) \right) + \left((e^{-10t}) \times \frac{d}{dt} (\cos(120\pi t)) \right) \right] \\ = 0.011 \left[(-10e^{-10t} \cos(120\pi t)) + (-120\pi e^{-10t} \sin(120\pi t)) \right] \\ = -0.011e^{-10t} [10\cos(120\pi t) + 120\pi \sin(120\pi t)] \text{ (amps)}$$

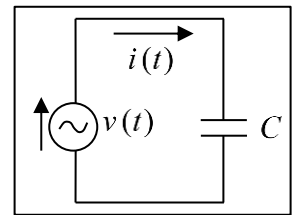
The term in square brackets can be reduced to a single, phase-shifted sine or cosine function. For example,

$$120\pi \sin(120\pi t) + 10\cos(120\pi t) = M \cos(120\pi t + \phi)$$

where

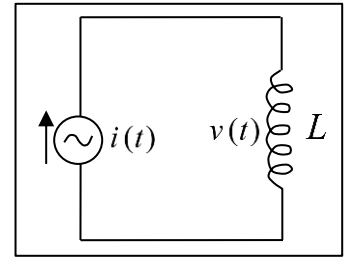
$$M = \sqrt{(120\pi)^2 + 10^2} \approx 377.1 \quad \text{and} \quad \phi = \tan^{-1}(-120\pi / 10) = -88.48^\circ = -1.544 \text{ (rad)}$$

$$\Rightarrow i(t) = -4.15e^{-10t} \cos(120\pi t - 1.544) \text{ (amps)}$$



Example 4:

Given: Current $i(t) = 5te^{-10t}$ (amps) is applied to an inductor with $L = 250$ (mh).



Find: $v(t)$, the voltage across the inductor.

Solution:

To find the voltage, we differentiate the current using the product and chain rules.

$$\begin{aligned}v(t) &= L \frac{di}{dt} = 0.25 \frac{d}{dt} (5te^{-10t}) = \frac{1}{4} \left(\frac{d}{dt} (5t) \right) (e^{-10t}) + \frac{1}{4} \left((5t) \frac{d}{dt} (e^{-10t}) \right) \\&= \frac{1}{4} (5e^{-10t}) + \frac{1}{4} (5t(-10e^{-10t})) = \frac{1}{4} (5e^{-10t} - 50te^{-10t}) \\&\Rightarrow \boxed{v(t) = \frac{5}{4} e^{-10t} (1 - 10t) \text{ (volts)}}\end{aligned}$$