

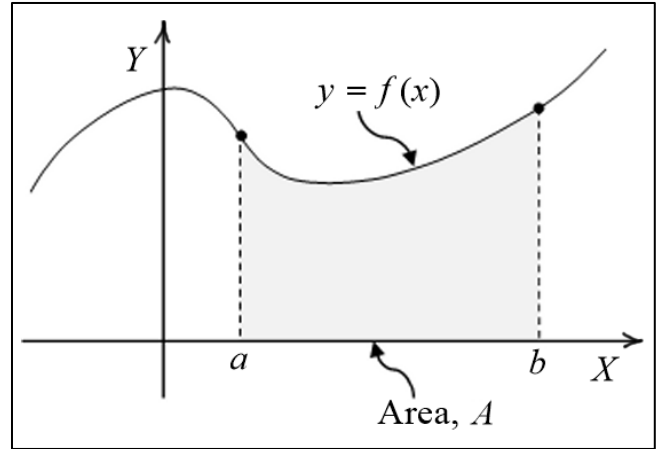
Elementary Engineering Mathematics

Introduction to the Integral of a Function

Definite Integral

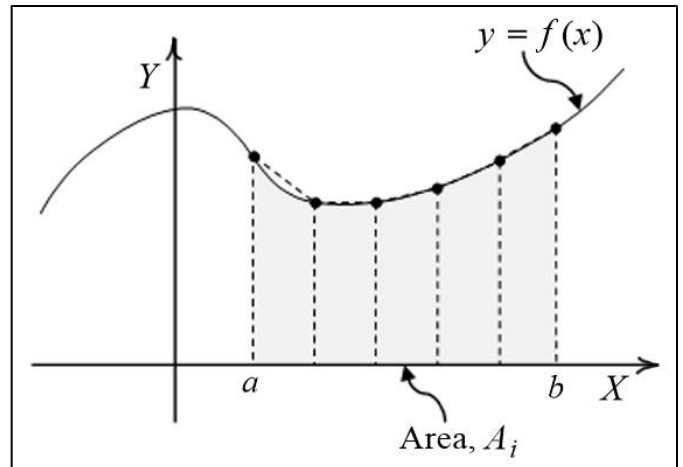
Consider a continuous function of a single variable $y = f(x)$. The **integral** of $f(x)$ from $x = a$ to $x = b$ is simply the **area under the curve** between those two points. The area is bounded on the underside by the X -axis. We write

$$A = \int_a^b f(x) dx \quad (1)$$

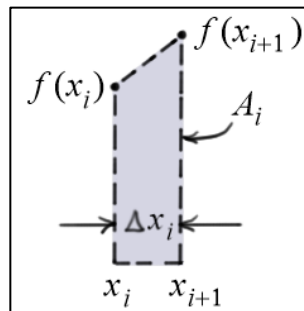


In this form, the integral is called a **definite integral**. It is a **number**, not a function. Later, we will define an **indefinite integral** which is itself a **function** of x .

One way to **estimate** the definite integral of a function is to break down the area into a finite number of **trapezoids** (as shown in the diagram) and **sum** the areas of all the trapezoids. As the increments Δx_i become smaller, Eq. (2) yields a more accurate estimate of the area A .



$$\begin{aligned} A &\approx \sum_i A_i \\ &\approx \sum_i \left[\frac{f(x_{i+1}) + f(x_i)}{2} \right] \Delta x_i \\ &\approx \sum_i \left[\frac{f(x_{i+1}) + f(x_i)}{2} \right] (x_{i+1} - x_i) \end{aligned} \quad (2)$$



So far, we considered the function $f(x)$ and the increments Δx_i to be **positive**. Consequently, the area A is **positive**. If, however, $f(x)$ is **negative** over this range the area will be **negative**. Clearly, if $f(x)$ takes on both **positive** and **negative** values, the resulting area could be **positive**, **negative**, or **zero**.

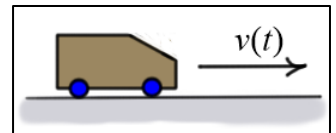
Given our current understanding of definite integrals, the following basic properties of integrals should seem reasonable.

	Property	Comment
1	$\int_a^b -f(x)dx = -\int_a^b f(x)dx$	if function values have opposite sign, the areas will also
2	$\int_b^a f(x)dx = -\int_a^b f(x)dx$	if increments have opposite sign, then areas will also
3	$\int_a^a f(x)dx = 0$	width of area is zero
4	$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$	total area = the sum of the areas
5	$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$	α is a constant
6	$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$	integral of a sum = the sum of the integrals

Note on units: The units of an integral are the same as the units of $f(x) \times \Delta x$.

Example 1:

Given: The displacement of a car as it moves with velocity $v(t)$ from time t_1 to t_2 is the integral of $v(t)$ over that period of time.

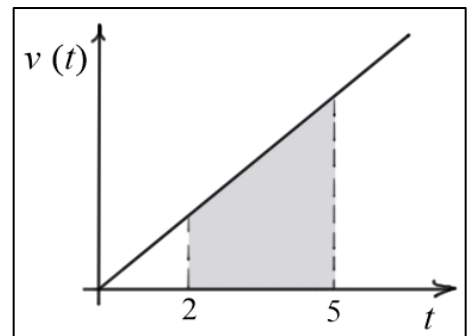


$$s = \int_{t_1}^{t_2} v(t)dt$$

The displacement can be positive, negative, or zero depending on how the sign of $v(t)$ changes over the interval t_1 to t_2 .

Find: Assuming the car has velocity $v(t) = 7.5t$ (ft/s²),

- (a) find the displacement of the car from 2 to 5 seconds;
- (b) find the total distance traveled from 2 to 5 seconds.



Solution:

(a) To calculate the shaded area, we can use a single trapezoid. The units of the result are the same as the units of $v(t) \times \Delta t \rightarrow (\text{ft/s}) \times \text{s} \rightarrow (\text{ft})$.

$$s = A = (5 - 2)(v(2) + v(5))/2 = 3(15 + 37.5)/2 = 78.75 \text{ (ft)}$$

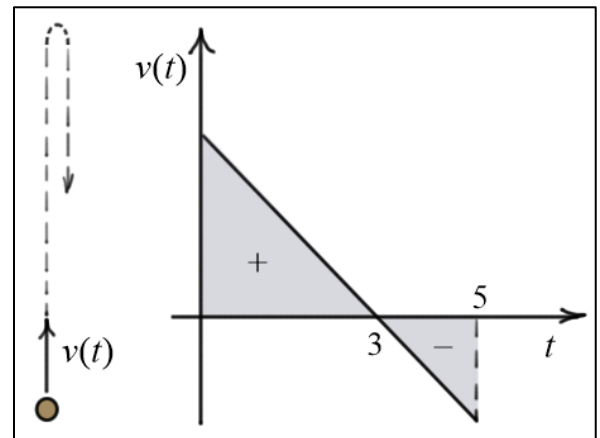
(b) The total distance traveled is also 78.75 (ft), because the velocity of the car is always positive from 2 to 5 seconds.

Example 2:

Given: The velocity of a ball for a certain period after it is thrown upward is

$$v(t) = 96.6 - 32.2t \text{ (ft/s)}$$

Find: (a) the vertical displacement of the ball from 0 to 5 seconds; and (b) the total distance traveled by the ball from 0 to 5 seconds.



Solution:

(a) The vertical displacement of the ball from 0 to 5 seconds is

$$s = \frac{1}{2}(3)(96.6) - \frac{1}{2}(5 - 3)(64.4) = 80.5 \text{ (ft)}$$

The downward movement of the ball is subtracted from the upward movement.

(b) The total distance traveled by the ball from 0 to 5 seconds is

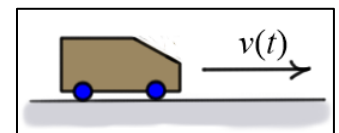
$$d = \frac{1}{2}(3)(96.6) + \frac{1}{2}(5 - 3)(64.4) = 209.3 \text{ (ft)}$$

The upward and downward movements of the ball are summed.

Example 3:

Given: The velocity of a car over the time interval from 0 to 5 seconds

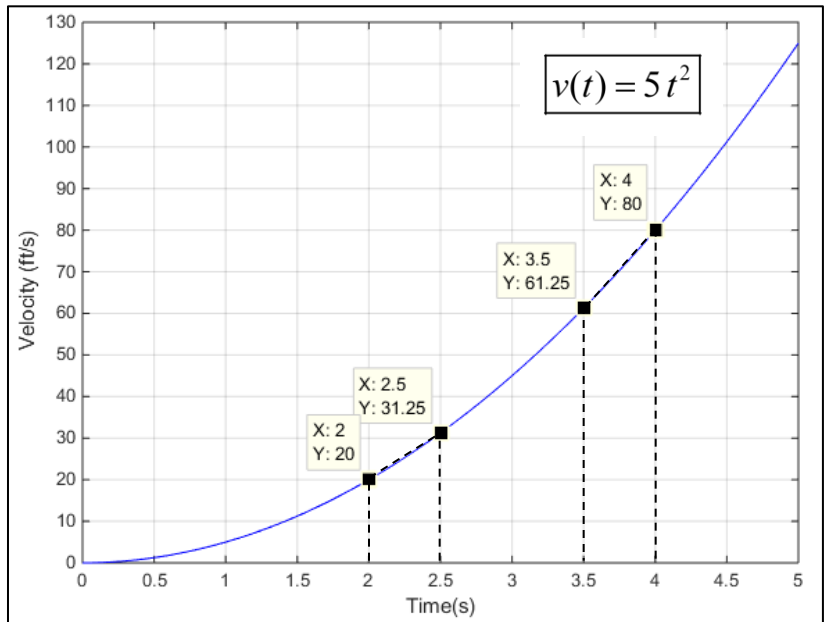
is $v(t) = 5t^2 \text{ (ft/s)}$.



Find: Estimate the distance traveled by the car from 2 to 5 seconds.

Solution:

Since the function is positive throughout the entire range of t , the total distance traveled is equal to the displacement. We can estimate the displacement by breaking up the area as shown in the diagram and the table below. **Two** of the areas are outlined on the plot with dashed lines.



Time, t	$f(t)$	Interval	f_{avg}	Δt
2	20	--		0.5
2.5	31.25	1	25.625	0.5
3	45	2	38.125	0.5
3.5	61.25	3	53.125	0.5
4	80	4	70.625	0.5
4.5	101.25	5	90.625	0.5
5	125	6	113.125	0.5
		Σ	391.25	

Since $\Delta t = 0.5$ (s) for all intervals, we have

$$s = A \approx \sum_{i=1}^6 \left[\frac{f(x_{i+1}) + f(x_i)}{2} \right] \Delta t_i = \sum_{i=1}^6 (f_{\text{avg}})_i \Delta t_i = \Delta t \sum_{i=1}^6 (f_{\text{avg}})_i \approx 0.5 \times 391.25 \approx 195.625 \text{ (ft)}$$

We will see later that the **actual displacement** is 195 (ft). Our current result is in **error** by about 0.32%.