

# Elementary Engineering Mathematics

## Applications of Integration in Elementary Dynamics

### Example 1: Acceleration profiles

**Given:** A car has an acceleration profile as shown. It's initial position and velocity are zero. ( $s(0) = v(0) = 0$ )

$$v(t) = \int a(t) dt \quad \text{and} \quad s(t) = \int v(t) dt$$

**Find:** (a) The velocity function  $v(t)$ ; and (b) the displacement function  $s(t)$  for the car for  $0 \leq t \leq 20$  (sec).

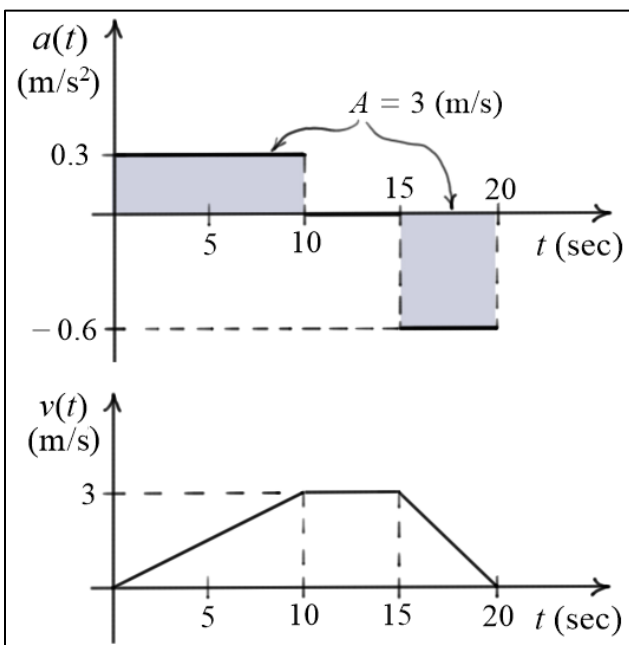
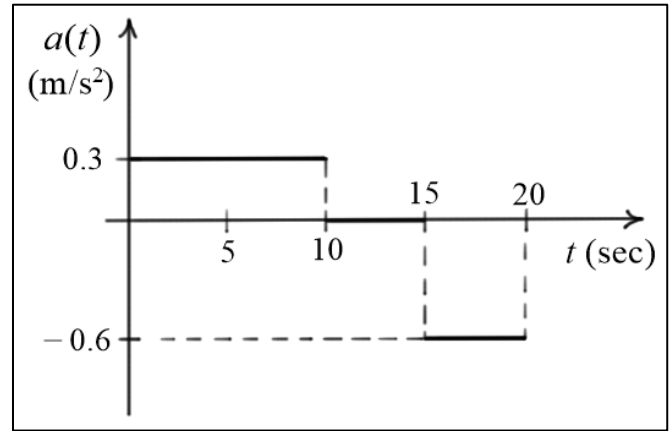
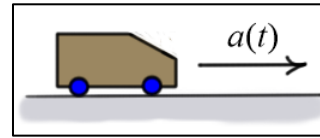
**Solution:**

(a) We can construct the velocity diagram easily from the acceleration diagram. When the acceleration is constant, the velocity varies linearly, and when the acceleration is zero, the velocity is constant.

$$v(t) = \int 0.3 dt = 0.3t + D = 0.3t \quad \text{for } 0 \leq t \leq 10 \quad (\text{recall that } v(0) = 0)$$

$$v(t) = \int 0 dt = D = v(10) = 3 \text{ (m/s)} \quad \text{for } 10 \leq t \leq 15$$

$$v(t) = \int -0.6 dt = -0.6t + D = -0.6t + 12 \quad \text{for } 15 \leq t \leq 20 \quad (v(15) = 3 \text{ (m/s)})$$



The **areas** under the acceleration profile give the **changes** in velocity over those periods of time

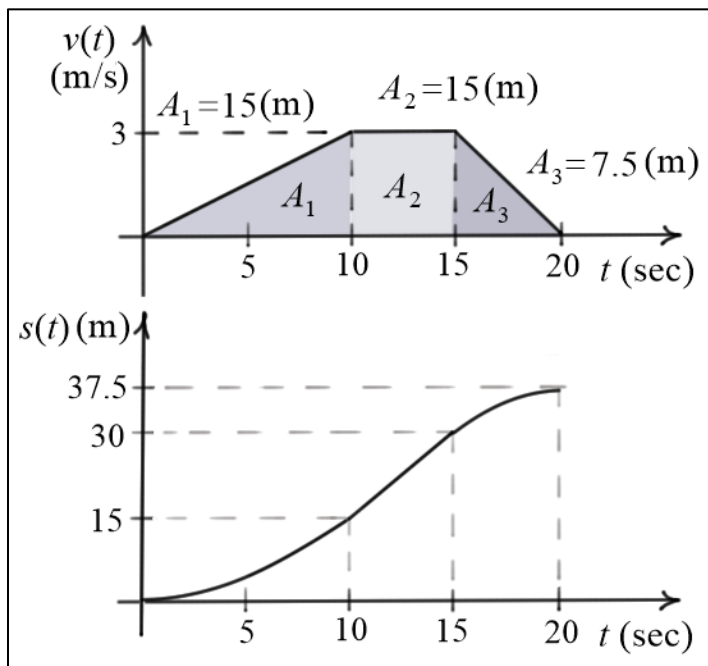
(b) The displacement function can now be derived from the velocity profile. When the velocity varies *linearly*, the displacement will vary *quadratically*, and when the velocity is *constant*, the displacement will vary *linearly*. The displacement changes are given by areas under the velocity function.

$$s(t) = \int 0.3t \, dt = 0.15t^2 + D = 0.15t^2 \quad \text{for } 0 \leq t \leq 10 \quad (\text{recall that } s(0) = 0)$$

$$s(t) = \int 3 \, dt = 3t + D = 3t - 15 \quad \text{for } 10 \leq t \leq 15 \quad (s(10) = 15 \text{ (m)})$$

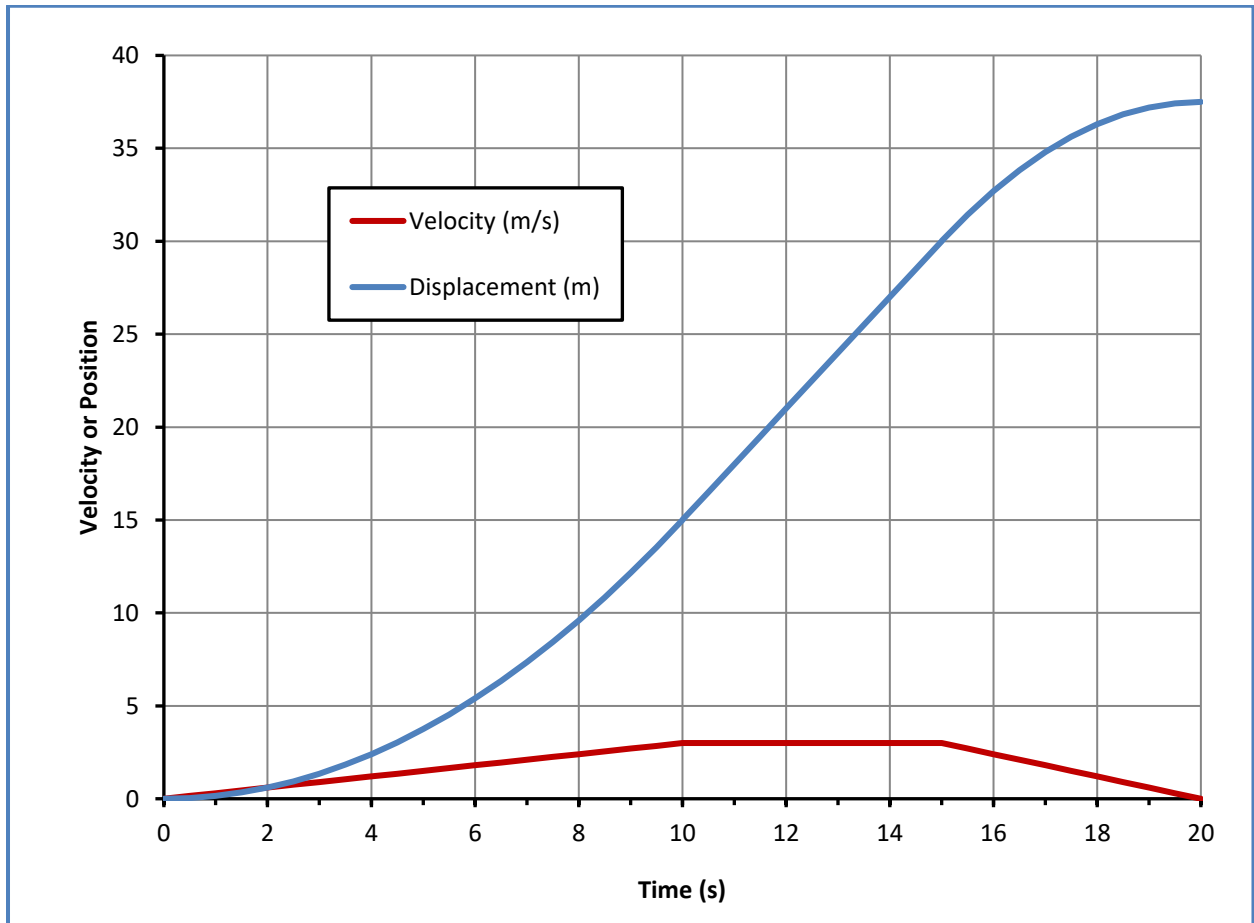
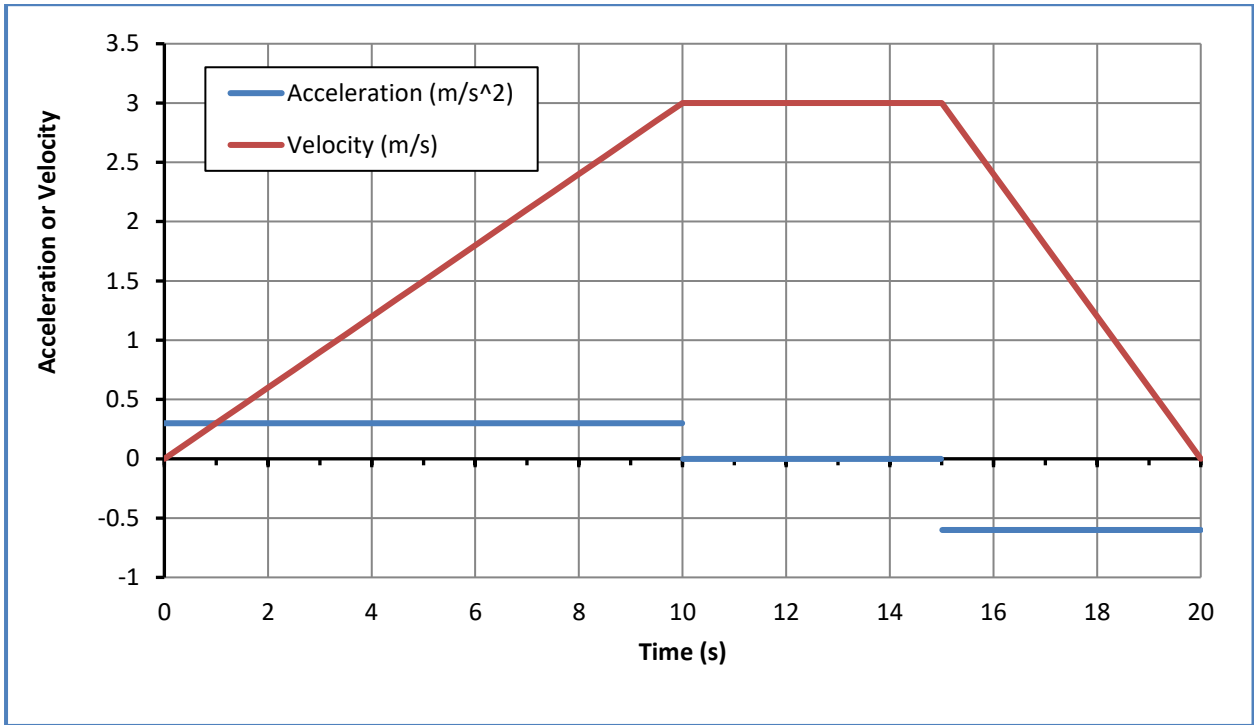
$$s(t) = \int -0.6t + 12 \, dt = -0.3t^2 + 12t + D$$

$$= -0.3t^2 + 12t - 82.5 \quad \text{for } 15 \leq t \leq 20 \quad (s(15) = 30 \text{ (m)})$$



The *areas* under the velocity profile give the *changes* in displacement over those periods of time

In this case, note that the acceleration profile is a *discontinuous* function, the velocity profile is a *piece-wise continuous* function, and the displacement profile is a *continuous* function. If we do not need the actual functions, we can still construct the velocity and displacement profiles by simply measuring areas under the acceleration and velocity profiles.



Example 2: Work done by a spring

Given: The work done by the spring force  $f(x)$  on the mass as it undergoes a downward displacement from  $A$  to  $B$  can be written as

$$W_{A \rightarrow B} = - \int_{x_A}^{x_B} f(x) dx$$

The work is **negative**, because the force is opposite the displacement of the mass. If the mass has a positive downward velocity during this displacement, the spring will **decrease** its velocity.

Find: The work done by a spring as it is stretched from  $x = 1$  (in) to  $x = 4$  (in), assuming (a) a linear spring with  $f(x) = 6x$  (lb); and (b) a non-linear hardening spring with  $f(x) = 3x^2$ . Assume  $x$  is measured in inches.

Solution:

(a) For the linear spring

$$W_{A \rightarrow B} = - \int_1^4 6x dx = - (3x^2) \Big|_1^4 = -3(16 - 1) = -45 \text{ (in-lb)}$$

(b) For the non-linear spring

$$W_{A \rightarrow B} = - \int_1^4 3x^2 dx = - (x^3) \Big|_1^4 = - (64 - 1) = -63 \text{ (in-lb)}$$

