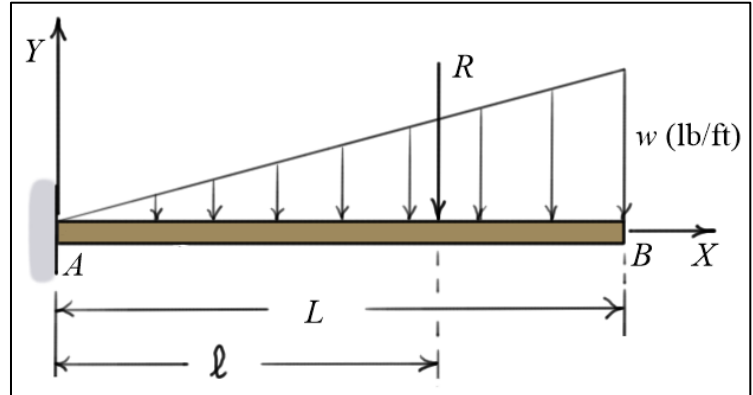


Elementary Engineering Mathematics

Applications of Integration in Statics and Mechanics of Materials

Example 1: Resultant of a distributed load

The diagram shows a cantilevered beam with a linearly varying load intensity. The maximum load intensity is w (lb/ft) at the right end of the beam. To calculate the support forces at A, the distributed load can be replaced by a single resultant load R acting at a distance ℓ from the left end.



The load R and distance ℓ are found by calculating the following integrals.

$$\boxed{R = \int_0^L wx \, dx} \quad \text{and} \quad \boxed{R \times \ell = \int_0^L (wx)x \, dx = \int_0^L wx^2 \, dx} \quad (1)$$

The first of Eqs. (1) equates the resultant *force* R with the *summation* (integral) of the *load intensity*, and the second one equates the *moment* of R about A with the *sum* (integral) of the *moments* of the of the *load intensity*.

Given: $L = 10$ (ft), $w = 100$ (lb/ft)

Find: (a) the resultant force R ; and (b) the distance ℓ that it acts from the support.

Solution:

(a) The resultant load is

$$\boxed{R = \int_0^{10} 10x \, dx = (5x^2) \Big|_0^{10} = 5 \times 100 = 500 \text{ (lb)}} \quad \text{or} \quad \boxed{R = \int_0^{10} 10x \, dx = \underbrace{\frac{1}{2} \times 10 \times 100}_{\text{area of the triangle}} = 500 \text{ (lb)}}$$

(b) The distance ℓ is

$$\boxed{500\ell = \int_0^{10} 10x^2 \, dx = \left(\frac{10}{3}x^3\right) \Big|_0^{10} \Rightarrow \ell = \frac{10 \times 10^3}{3 \times 500} = \frac{2}{3}(10) = 6.\bar{6} \text{ (ft)}}$$

For a linearly varying load (starting at zero), the resultant acts $\frac{2}{3}$ of the way along the distributed load. If the load is w (lb/ft) at the wall and zero at the end of the beam, the resultant would be located $\frac{1}{3}$ of the way along the load.

Example 2: Uniformly distributed load

Given: $L = 10$ (ft), $w = 100$ (lb/ft)

Find: (a) the resultant force R ; and (b) the distance ℓ that it acts from the support.

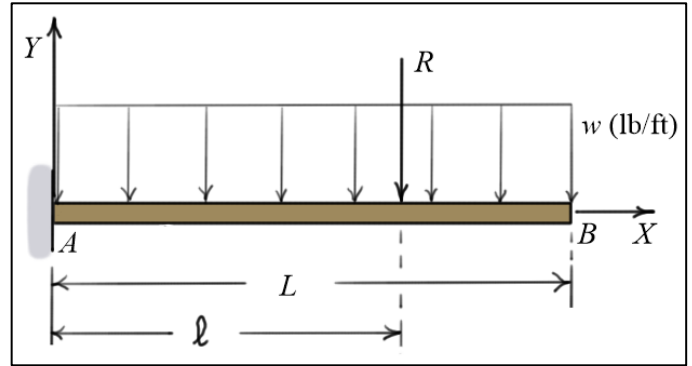
Solution:

(a) The resultant load is

$$R = \int_0^{10} 100 dx = 10 \times 100 = 1000 \text{ (lb)}$$

(b) The distance ℓ is

$$1000 \ell = \int_0^{10} 100x dx = \frac{1}{2} \times 10 \times 1000 \Rightarrow \ell = \frac{5000}{1000} = \frac{1}{2}(10) = 5 \text{ (ft)}$$



So, for a uniformly distributed load, the resultant acts $\frac{1}{2}$ of the way along the load.

Example 3:

The diagram of the internal shearing force for the simply supported beam with a concentrated load is shown. The internal bending moment is related to the shearing force by the equation

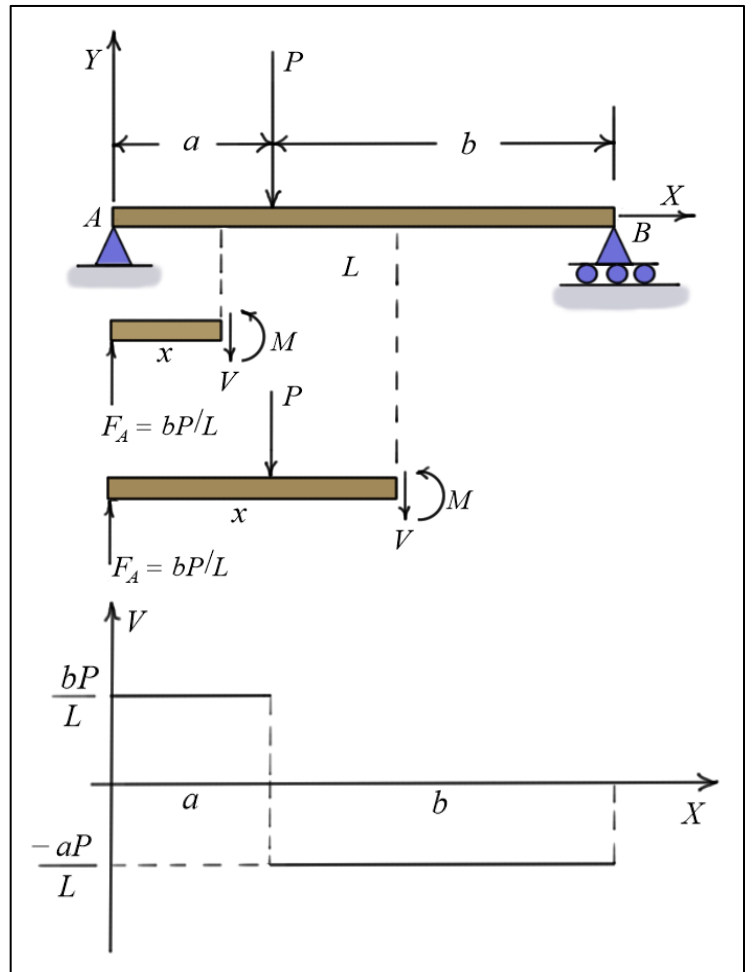
$$M(x) = \int V(x) dx$$

Given: $P = 100$ (lbs), $L = 5$ (ft),

$a = 3.5$ (ft), $b = 1.5$ (ft),

and $M(0) = M(L) = 0$

Find: (a) moment diagram for the beam; and (b) M_{\max} the maximum bending moment in the beam.



Solution:

- (a) We can construct the moment diagram from the shear diagram. Where the shear is *constant*, the moment varies *linearly* with x .

$$V_a = bP/L = 1.5 \times 100/5 = 30 \text{ (lb)} \quad \text{and} \quad V_b = -aP/L = -3.5 \times 100/5 = -70 \text{ (lb)}$$

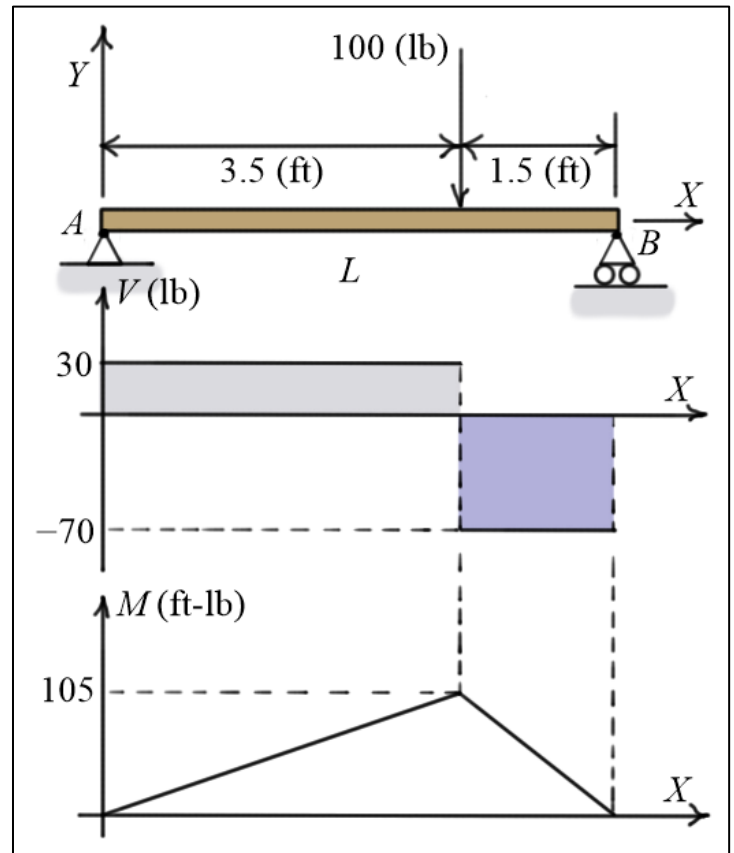
$$M(x) = \int 30 dx = 30x + D = 30x \quad \text{for } 0 \leq x \leq 3.5 \text{ (recall that } M(0) = 0)$$

$$M(x) = \int -70 dx = -70x + D = -70x + 350 \quad \text{for } 3.5 \leq x \leq 5 \text{ (} M(3.5) = 105 \text{ (ft-lb))}$$

- (b) The maximum bending moment occurs at the concentrated load. It is the area under the shear diagram from 0 to 3.5 feet.

$$M_{\max} = 30 \times 3.5 = 105 \text{ (ft-lb)}$$

Note that the area under the shear diagram from 3.5 to 7 feet is the negative of this value, so the bending moment at both ends of the beam are zero.



Example 4:

Given: $L = 10$ (ft), $w = 100$ (lb/ft),

$$M(0) = M(L) = 0, \text{ and}$$

$$V(x) = 500 - 100x \text{ (lb)} \quad (0 \leq x \leq L)$$

Find: (a) bending moment diagram; and
(b) the maximum bending moment.

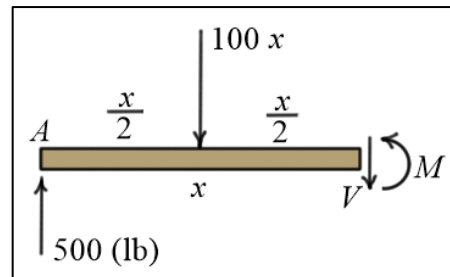
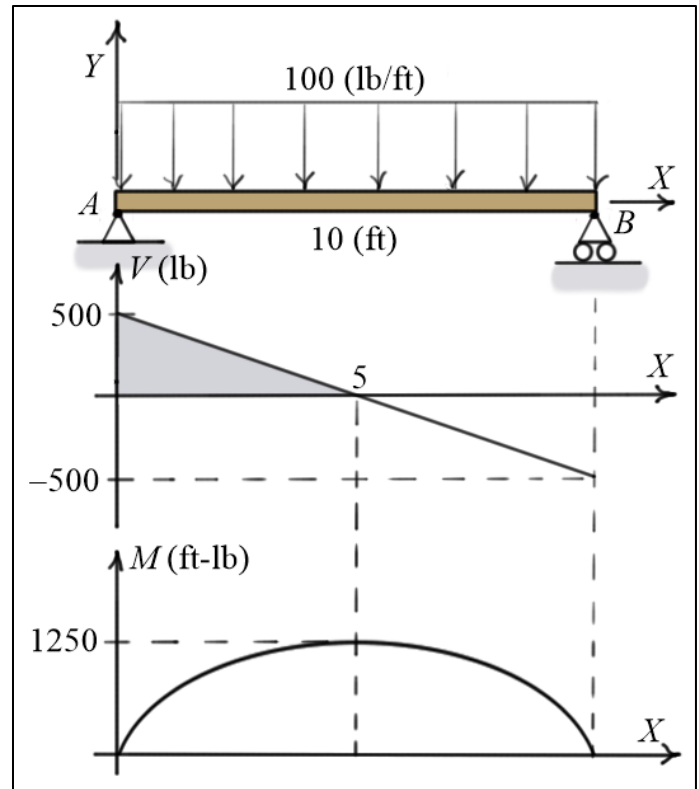
Solution:

- (a) Again, we can construct the moment diagram from the shear diagram. Where the shear varies *linearly*, the moment varies *quadratically* with x .

$$\begin{aligned} M(x) &= \int (500 - 100x) dx \\ &= 500x - 50x^2 + D \quad (M(0) = 0) \\ &= 500x - 50x^2 \text{ (ft-lb)} \end{aligned}$$

- (b) The maximum bending moment occurs at the midpoint of the beam. It is the *area under the shear diagram* from 0 to $L/2$.

$$M_{\max} = \frac{1}{2} \times 5 \times 500 = 1250 \text{ (ft-lb)}$$



Example 5:

Given: $L = 10$ (ft), $w = 100$ (lb/ft), $M(L) = 0$,

$$V(x) = 1000 - 100x \text{ (lb)} \quad (0 \leq x \leq L)$$

Find: (a) bending moment diagram; and
(b) the maximum bending moment.

Solution:

(a) In this case, the shear varies *linearly* with x , so the moment will vary *quadratically* with x . Given $M(10) = 0$, we find

$$\begin{aligned} M(x) &= \int (1000 - 100x) dx \\ &= 1000x - 50x^2 + D \\ &= -5000 + 1000x - 50x^2 \text{ (ft-lb)} \end{aligned}$$

(b) The maximum moment occurs at the left end of the beam and is equal to the area under the shear diagram from 0 to L . Why?

$$M_{\max} = \frac{1}{2} \times 10 \times 1000 = 5000 \text{ (ft-lb)}$$

