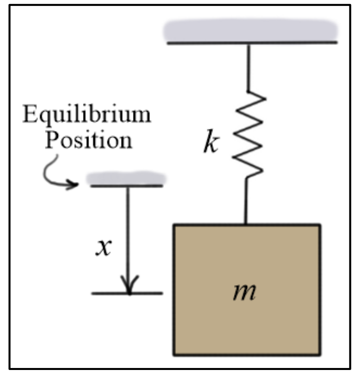
Elementary Engineering Mathematics
Equations Sheet #4 – Spring-Mass-Damper Systems

1. Spring-Mass System (no damping)



Initial Conditions: $x(0) = x_0$ $v(0) = v_0$

Natural Frequency: $\omega = \sqrt{\frac{k}{m}}$

Displacement: $x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$

Velocity: $v(t) = v_0\cos(\omega t) - x_0\omega\sin(\omega t)$

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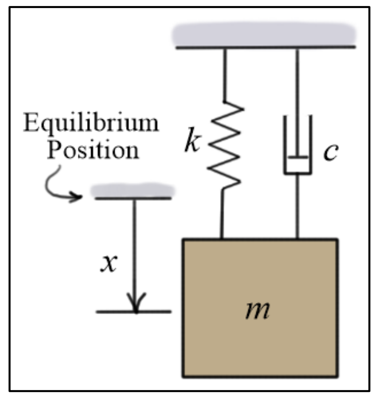
2. Spring-Mass-Damper System



Critical Damping Coefficient: $c_c = 2m\sqrt{k/m}$

Over-damped Motion: $c > c_c$

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$
$$v(t) = A\lambda_1 e^{\lambda_1 t} + B\lambda_2 e^{\lambda_2 t}$$

$$\begin{Bmatrix}\lambda_1 \\ \lambda_2\end{Bmatrix} = \begin{Bmatrix} -\left(\frac{c}{2m}\right) + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{Bmatrix}$$

Coefficients: (A, B) found by solving the simultaneous equations: $\begin{matrix} A + B = x_0 \\ \lambda_1 A + \lambda_2 B = v_0 \end{matrix}$

Under-Damped Motion: $c < c_c$

Frequency: $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

Displacement: $x(t) = e^{-(c/2m)t}\left(A\sin(\omega_d t) + B\cos(\omega_d t)\right)$ $\quad A = \left[v_0 + \left(\frac{c}{2m}\right)x_0\right]/\omega_d$, $B = x_0$

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3. Decay/Growth Rate

$$\alpha = \frac{\ln(x_2/x_1)}{t_2 - t_1}$$