

Elementary Engineering Mathematics

Equations Sheet #8 – Differential Equations for a Spring-Mass-Damper System

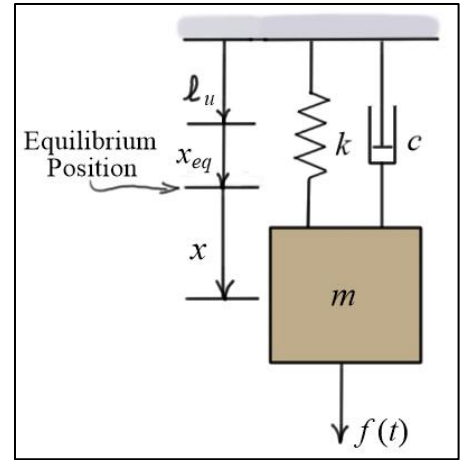
1. Differential Equation of Motion

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

2. Free Response ($f(t) = 0$)

(a) Characteristic Equation: $ms^2 + cs + k = 0$

(b) Form of Solution depends on type of roots. Coefficients found by applying initial conditions.



Case	Type of Roots	Type of Motion	Form of Solution
1	Real, unequal	Over-damped	$x(t) = Ae^{s_1 t} + Be^{s_2 t}$
2	Real, equal	Critically damped	$x(t) = Ae^{st} + Bte^{st}$
3	Complex conjugates	Under-damped	$x(t) = e^{-(c/2m)t} [A \sin(\omega_d t) + B \cos(\omega_d t)]$ $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

3. Forced Response ($f(t) \neq 0$)

(a) Solution is the sum of the homogeneous solution and a particular solution: $x(t) = x_H(t) + x_P(t)$.

(b) Homogeneous solution has the **form** of the free response.

(c) The **forms** of some particular solutions are shown in the table below.

(d) Coefficients are found by applying initial conditions.

	$f(t)$	Form* of $x_P(t)$
constant	a_0	$B_0 t^n$
linear	$a_1 t + a_0$	$(B_1 t + B_0) t^n$
quadratic	$a_2 t^2 + a_1 t + a_0$	$(B_2 t^2 + B_1 t + B_0) t^n$
exponential	$a e^{\beta t}$	$(B_1 e^{\beta t}) t^n$
sine or cosine	$a \sin(\omega t)$ or $a \cos(\omega t)$	$[B_1 \sin(\omega t) + B_2 \cos(\omega t)] t^n$
exponential-sine or exponential-cosine product	$a e^{\beta t} \sin(\omega t)$ or $a e^{\beta t} \cos(\omega t)$	$e^{\beta t} [B_1 \sin(\omega t) + B_2 \cos(\omega t)] t^n$

* The **exponent** n is the **smallest, non-negative integer** so every term in $x_P(t)$ is **different** from every term in $x_H(t)$. That is, $n = 0$ unless the same type of term appears in $x_H(t)$.