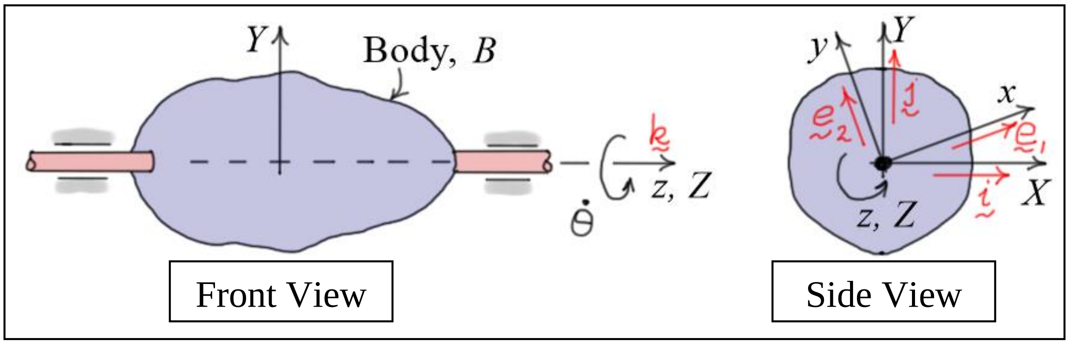
Intermediate Dynamics
Simple Angular Motion

Simple Angular Velocity

The rigid body B shown in the diagram below rotates about the Z-axis. The XYZ reference frame is a fixed frame, while the xyz reference frame is fixed in (and rotates with) the body. The XYZ reference frame is represented by the unit vector set $R:(\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k})$, and the xyz reference frame is represented by the unit vector set $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{k})$. Note that ***each*** unit vector ***set*** is a ***right-handed*** set, that is $\underset{\sim}{i}\times\underset{\sim}{j}=\underset{\sim}{k}$ and $\underset{\sim}{e}_1\times\underset{\sim}{e}_2=\underset{\sim}{k}$.



The unit vectors fixed in the body B can be ***differentiated*** by using the concept of ***angular velocity***. It can be shown that

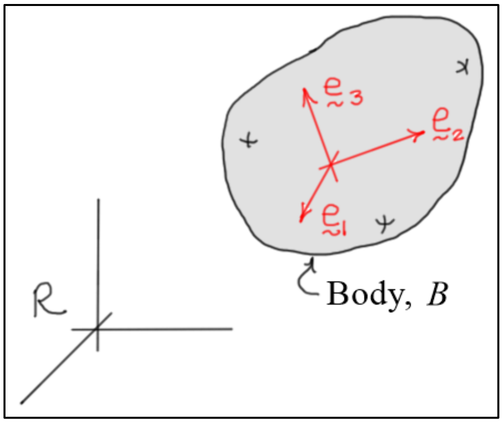
$$\boxed{\frac{{}^R d\underset{\sim}{e}_i}{dt} = {}^R\underset{\sim}{\omega}_B\times\underset{\sim}{e}_i}\qquad (i=1,2)$$

where $\frac{{}^R d\underset{\sim}{e}_i}{dt}$ represents the ***derivative*** of the unit vector $\underset{\sim}{e}_i$ in the ***reference frame*** R, and ${}^R\underset{\sim}{\omega}_B=\dot{\theta}\underset{\sim}{k}$ is the ***angular velocity*** of the body B in the ***reference frame*** R.

Aside:

$$\begin{aligned}
\frac{{}^R d\underset{\sim}{e}_1}{dt} &= \frac{{}^R d}{dt}(C_\theta\underset{\sim}{i}+S_\theta\underset{\sim}{j})\\
&=\dot{\theta}(-S_\theta\underset{\sim}{i}+C_\theta\underset{\sim}{j})\\
&=\dot{\theta}\underset{\sim}{e}_2\\
&=\dot{\theta}(\underset{\sim}{k}\times\underset{\sim}{e}_1)\\
&={}^R\underset{\sim}{\omega}_B\times\underset{\sim}{e}_1
\end{aligned}$$

Differentiation of Unit Vectors – General Case



Consider now a rigid body B moving in three-dimensional space. In general, given a set of unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ fixed in B, it can be shown that

$$\boxed{\frac{{}^{R}d\underset{\sim}{e}_i}{dt} = {}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{e}_i} \qquad (i = 1,2,3)$$

where, as before, $\frac{{}^{R}d\underset{\sim}{e}_i}{dt}$ represents the derivative of the unit vector $\underset{\sim}{e}_i$ in the reference frame R, and ${}^{R}\underset{\sim}{\omega}_B$ is the angular velocity of the body B in the reference frame R. What is needed now is a means of calculating ${}^{R}\underset{\sim}{\omega}_B$, if the body doesn't have simple angular motion.

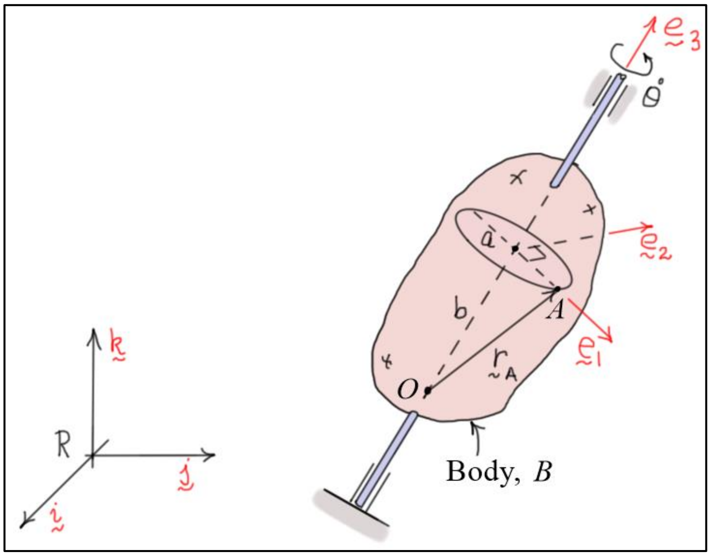
Simple Angular Acceleration

The angular acceleration of B in R is found by ***differentiating*** the angular velocity vector. That is,

$$\boxed{{}^{R}\underset{\sim}{\alpha}_B = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_B\right) = \ddot{\theta}\,\underset{\sim}{k}}$$

Kinematics of Fixed Axis Rotation

Consider the rigid body B shown in the diagram below. The fixed reference frame is represented by the unit vector set $R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$, and the rotating reference frame is represented by the unit vector set $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$. All points of B travel in a circular path around the fixed axis. The ***velocity*** and ***acceleration*** of any point within the body can be determined by differentiating (with respect to time) its position vector $\underset{\sim}{r}_A$ relative to any point on the fixed axis.



For example, the velocity of point A can be calculated as follows

$$\underset{\sim}{v}_A = \frac{{}^R d}{dt}(a\underset{\sim}{e}_1 + b\underset{\sim}{e}_3) = a\frac{{}^R d\underset{\sim}{e}_1}{dt} + b\frac{{}^R d\underset{\sim}{e}_3}{dt}$$
$$= a({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{e}_1) + b({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{e}_3)$$
$$= {}^R\underset{\sim}{\omega}_B \times (a\underset{\sim}{e}_1 + b\underset{\sim}{e}_3)$$
$$= {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_A$$

Performing the cross product in the last equation gives $\boxed{\underset{\sim}{v}_A = a\dot{\theta}\underset{\sim}{e}_2}$. Note the velocity is ***tangent*** to the ***circular path***. Similarly, the acceleration of A can be calculated as follows

$${}^R\underset{\sim}{a}_A = \frac{{}^R d}{dt}({}^R\underset{\sim}{v}_A) = \frac{{}^R d}{dt}({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_A)$$
$$= ({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_A) + ({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_A)$$

Performing the operations in this last equation gives $\boxed{{}^R\underset{\sim}{a}_A = -a\dot{\theta}^2\underset{\sim}{e}_1 + a\ddot{\theta}\underset{\sim}{e}_2}$. Note the acceleration has components both ***normal*** and ***tangent*** to the circular path.